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MATHEMATICAL ANALYSIS

Differentiation and Integration

$$\left(\frac{\partial x}{\delta u}+\frac{\partial x}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x}+\left(\frac{\partial x}{\partial v}+\frac{\partial x}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$\left(\frac{\partial y}{\partial u}+\frac{\partial y}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x}+\left(\frac{\partial y}{\partial v}+\frac{\partial y}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x}$$

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MATHEMATICAL ANALYSIS

Differentiation and Integration

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FOREWORD

THE present volume of the series in Pure and Applied Mathematics is devoted to two basic operations of mathematical analysis – differentiation and integration. It discusses the complex of problems directly connected with the operations of differentiation and integration of functions of one or several variables, in the classical sense, and also elementary generalizations of these operations. Further generalizations will be given in subsequent volumes of the series, volumes devoted to the theory of functions of real variables and to functional analysis.

Together with an earlier volume in the series, volume 69, L. A. Lyusternik and A. R. Yanpol'skii, *Mathematical Analysis (Functions, Limits, Series, Continued Fractions)*, the present one includes material for a course of mathematical analysis, which is treated in a logically connected manner, briefly and without proofs, but with many examples worked in detail.

Chapter I "The differentiation of functions of one variable" (authors: L. A. Lyusternik and R. S. Guter) and Chapter II "The differentiation of functions of n variables" (author: L. A. Lyusternik) contain a discussion of derivatives and differentials, their properties and their application in investigating the behaviour of functions, Taylor's formula and series, differential operators and their elementary properties, stationary points, and also the extrema of functions of one variable (author: I. G. Aramanovich) and of n variables (author: I. L. Raukhvarger).

Chapter III "Composite and implicit functions of n variables" (authors: R. S. Guter and I. L. Raukhvarger) contains a discussion of general problems of the theory of functions of n variables in connection with differentiation. Here belong composite and implicit functions, the representation of functions in the form of superpositions, etc. A separate section (author: V. A. Trenogin) is devoted to Newton's diagram.

In view of the particular importance of functions of two and three variables in their application to problems of analysis, they are

separated out to form a chapter on their own, Chapter IV "Systems of functions and curvilinear coordinates in a plane and in space" (author: M. I. Skanavi), where a detailed description is given of the properties of mappings of one region into another (in particular, affine mappings) and of different systems of curvilinear coordinates. This chapter (as also Chapter VII) is based on the book by A. F. Bermant [2].

Chapter V "The integration of functions" (authors: R. S. Guter, I. L. Raukhvarger and A. R. Yanpol'skii) contains a discussion of the properties of integrals, methods of integrating elementary functions and the application of integrals to geometrical and mechanical problems.

Certain generalizations of the concept of an integral are dealt with in Chapter VI, "Improper integrals; integrals depending on a parameter; the integral of Stieltjes" (authors: I. G. Aramanovich, R. S. Guter and I. L. Raukhvarger). Here, a detailed account is given of improper integrals and their properties, the concept of Stieltjes' integral is given, and also of integrals and derivatives of fractional order.

In Chapter VII, "The transformation of differential and integral expressions" (author: M. I. Skanavi) the classification is given of various cases of transformation of the expression named in the heading of the chapter, general rules for the change of variables in the differential and the integral expressions are laid down, and a summary is given of expressions for the basic differential operations (gradient, divergence, curl, Laplacian) in the transformation of rectangular cartesian coordinates to various curvilinear orthogonal coordinates (compiled by V. I. Bityutzkov). Here also the discussion of surface integrals is systematized, and Green's formulae with various generalizations are given.

In the appendixes there are given tables of derivatives of the first and the nth order of elementary functions, the expansion of functions into power series and of integrals (indefinite, definite and multiple). Tables are also to be found of special functions, functions defined by means of integrals of elementary functions (elliptic integrals, integral functions, Fresnel integrals, gamma-functions, etc.).

NOTATION

$[x]$	operator of multiplication by the argument
$C = C[X]$	the class of functions $f(x)$ defined and continuous in the set X
$C_n = C_n[X]$	the class of functions $f(x)$ defined and continuously differentiable n times in the set X
C_G	the class of functions $f(X)$ defined and continuous in the region G
$C_1 = C_{1,G}$	the class of functions $f(X)$, all of whose first partial derivatives in the region G are defined and continuous
$C_n = C_{n,G}$	the class of functions, all of whose nth partial derivatives in the region G are defined and continuous
$\Delta_h f(x_0)$	the increment of the function at the point x_0
$\Delta_{h_i}^{x_i}(X)$	the partial increment of a function
$\Delta_h^2 f(x_0)$, $\Delta_h^n f(x_0)$	the second and the nth difference at the point x_0
$y', \dfrac{dy}{dx}, \dfrac{df(x)}{dx}, \dfrac{d}{dx}f(x), Df(x), \dot{u}(t)$	the derivative of a function
$f'_-(x_0), f_-^{(s)}(x_0)$	the first and the sth left-hand derivative at the point x_0
$f'_+(x_0), f_+^{(s)}(x_0)$	the first and the sth right-hand derivative at the point x_0
$f''(x_0) \equiv \left.\dfrac{d^2y}{dx^2}\right\rvert_{x=x_0}$	the second derivative of a function at point x_0
$f^n(x_0) \equiv \left.\dfrac{d^n f(x)}{dx^n}\right\rvert_{x=x_0}$	the nth derivative of a function at point x_0
$\tilde{f}''(x_0), \tilde{f}^{(n)}(x_0)$	the second and the nth differential derivatives of a function at the point x_0
$f^{(')}(x_0), f^{('')}(x_0), f^{(n)}(x_0)$	Schwartzian derivatives at the point x_0
$\dfrac{\partial}{\partial x_i} f(X^0)$	the partial derivative at the point X^0

$P_n(D)$	differential polynomial (polynomial of the operator D)
$\dfrac{D(y_1, y_2, \ldots, y_n)}{D(x_1, x_2, \ldots, x_n)}$, $\dfrac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$	Jacobian
dy	the first differential
d^2y, d^ny	the second and the nth differential
$df(x_0)$, $df(x_0, h)$	the first differential at the point x_0
$d^2f(x_0)$, $d^2f(x_0, h)$, $d^nf(x_0)$, $d^nf(x_0, h)$	the second and the nth differential at the point x_0
Ax	the operator, defined in the set X of elements x
$D = \dfrac{d}{dx}$, $D^n = \dfrac{d^n}{dx^n}$	the operator of differentiation
$df(X^0, H)$	the differential of the operator $f(X)$ at the point X^0
$\Delta\dfrac{x_i}{h_i}$	the operator of partial increment in x_i
$D_i = \dfrac{\partial}{\partial x_i}$	the operator of partial differentiation in x_i
Δ, ∇^2	Laplace's operator
$\operatorname{grad} f(X^0)$	the gradient of a function $f(X)$ at the point X^0
$\operatorname{grad} z$	the gradient of the function z
$\operatorname{div} \mathbf{a}$	divergence of the vector $\mathbf{a}$
$\operatorname{rot} \mathbf{a}$	curl of the vector $\mathbf{a}$
$\varrho(X, Y)$	distance between the points $X(x_1, x_2, \ldots, x_n)$ and $Y(y_1, y_2, \ldots, y_n)$
$\lVert X \rVert$	the norm of a vector
h_u, h_v, H_u, H_v, H_w	differential parameters of the first order
E, F, G	Gauss' coefficients
l_u, l_v, L_u, L_v, L_w	Lamé's coefficients
ds	an element of length
dq, Δq	an element of area for a plane
$d\sigma$, $\Delta\sigma$	an element of area for a surface
dv	an element of volume
$(R)\displaystyle\int_a^b f(x)\,dx$	Riemann's integral
$\displaystyle\int_a^b f(x)\,d\varphi(x)$	Stieltjes' integral
$Y = f(X)$	mapping from E_n into E_m

$S(X^0, r)$	a sphere of radius r and centre at point X^0
$\mathcal{S}$	the sum sign of several analogous expressions
$J[y(x)]$	functional of $y(x)$
$B(\alpha, \beta)$	Euler's beta function
$\Gamma(\alpha)$	Euler's gamma function
$\Pi(x)$	pi-function
$\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$	
$\Psi(x)$	the logarithmic derivative of the pi-function
$\varphi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$	
$\mathrm{Ei}\, x$	exponential integral
$\mathrm{li}\,(x)$	logarithmic integral
$\mathrm{Si}\,(x)$	sine integral
$\mathrm{Ci}\,(x)$	cosine integral
$\mathrm{erf}\,(x)$, $\Phi(x)$	integrals in probability theory
$F(k, \varphi)$	elliptic integral of the first kind
$E(k, \varphi)$	elliptic integral of the second kind
$\mathbf{K} = F\left(k, \dfrac{\pi}{2}\right)$	complete elliptic integral of the first kind
$\mathbf{E} = E\left(k, \dfrac{\pi}{2}\right)$	complete elliptic integral of the second kind
$S(x)$, $S^*(x)$	Fresnel's sine-integrals
$C(x)$, $C^*(x)$	Fresnel's cosine-integrals
$\mathbf{C}$	Euler's constant

CHAPTER I

THE DIFFERENTIATION OF FUNCTIONS OF ONE VARIABLE

THE basic operations of mathematical analysis are operations on functions which are mutually inverse—differentiation and integration. This chapter is devoted to the operation of the differentiation of functions of one variable. The concepts of function, limiting process, the properties of continuous functions and similar topics, which precede the study of the operation of differentiation in analysis courses, are dealt with in volume 69 of the series in Pure and Applied Mathematics called *Mathematical Analysis (Functions, Limits, Series, Continued Fractions)* edited by L. A. Lyusternik and A. R. Yanpol'skii (Pergamon Press, Oxford, 1965).

§ 1. Derivatives and Differentials of the First Order

1. Suppose that the function of one variable, $y = f(x)$, is defined in the set X, which is a line segment or an open interval, or a semi-open interval. Unless the contrary is stated, the points x_0, $x_0 + h$ are assumed to be interior points of the set X.

The derivative of function $y = f(x)$, defined in the set X, at point $x = x_0 \in X$ is the name given to the limit of the ratio

$$\frac{\Delta y}{h} = \frac{f(x_0 + h) - f(x_0)}{h}$$

(of the increment Δy of function y to the increment h of the argument x) when $h \to 0$, if this limit exists. The derivative of the function $f(x)$ at the point $x = x_0$ is denoted by $f'(x_0)$. Thus

$$f'(x_0) = \lim_{h\to 0} \frac{\Delta y}{h} = \lim_{h\to 0} \frac{\Delta_h f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \quad (1.1)$$

where $x_0 + h \in X$, $\Delta hf(x_0)$ is the increment of the function $f(x)$ at the point $x = x_0$.

If the point x_0 is the end-point of a segment, the limit (1.1) determining the derivative is looked upon as one-sided; a right one, when the point x_0 is the left end of the segment; and a left one, when the point x_0 is its right end.

Geometrically, the derivative $f'(x_0)$ represents the tangent of the angle which the tangent at the point $x = x_0$ to the curve $y = f(x)$ makes with the x-axis.

The ratio $\Delta y/h$ might turn out to be infinitely large in the vicinity of the point $x = x_0$. If, for $h \neq 0$, the ratio $\Delta y/h$ tends to infinity *with a definite sign*, it is said that the function has *an infinite derivative* $f'(x) = +\infty$, or $f'(x) = -\infty$ at the given point. Geometrically, this means, that the curve $y = f(x)$ has a tangent parallel to the axis Oy at the point $x = x_0$. For example, the curve $y = \sqrt[3]{x}$ has a vertical tangent at the point $x_0 = 0$, since

$$y'\Big|_{x=0} = \tfrac{1}{3}x^{-\frac{2}{3}}\Big|_{x=0} = +\infty.$$

The function $f(x)$ is said to be *differentiable* at the point $x = x_0$, if it has a finite derivative at the point x_0. If $f(x)$ is differentiable at every point x of the set X, it is said to be *differentiable in the set* X; its derivative $f'(x)$ is a function of the point x of the set X – *the derived function*. The operation (or operator) of differentiation relates to the function $f(x)$ its derivative $f'(x)$; the initial function $f(x)$ is called the *antiderivative* with respect to its derivative.

EXAMPLE 1. For the function $f(x) = C$ (C is constant) the derivative equals zero:

$$(C)' = 0.$$

EXAMPLE 2. If $f(x) = x$, the derivative equals 1:

$$(x)' = 1.$$

EXAMPLE 3. The derivative of the function $f(x) = \sin x$ equals

$$(\sin x)' = \lim_{h\to 0} \frac{\sin (x+h) - \sin x}{h} = \cos x.$$

To denote the derivative of the function $y = f(x)$ the following symbols are also used: y', dy/dx, $df(x)/dx$, $Df(x)$.

If the argument t denotes time, the derivative of the function $u(t)$ is also denoted by the symbol $\dot{u} = \dot{u}(t)$ *(Newton's notation)*. The derivative $u(t)$ means the rate of change of the quantity $u(t)$ at the

moment t. For example, if $x(t)$ is the distance traversed by a point moving in a straight line in time t, then $\dot{x}(t)$ means the velocity of the point at a moment of time t.

The operation of differentiation relates to the sum of functions the sum of their derivatives and preserves a constant coefficient (the linear property):

$$[f(x) + \varphi(x)]' = f'(x) + \varphi'(x), \tag{1.2}$$

$$[cf(x)]' = cf'(x) \qquad (c = \text{const}). \tag{1.3}$$

The formulae for the derivatives of a product and a quotient of the functions $u(x)$ and $v(x)$ have the form:

$$(uv)' = u'v + uv', \tag{1.4}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}. \tag{1.5}$$

These formulae may be written down in the form

$$u'v + uv' = (uv)',$$

$$u'v - uv' = v^2(uv^{-1})'.$$

2. One of the fundamental rules of differentiation is the rule of differentiating a composite function.

Suppose the function $y = f(x)$ is defined in the set X and Y is the set of its possible values. Let us consider the function $z = \varphi(y)$ defined in the set Y, i.e. a function, whose argument is in its turn a function of one variable. Then, to each value x of the set X, there corresponds some definite value z, i.e. the variable z is a function of x. It can be written down in the form

$$z = \varphi[f(x)].$$

Such an expression is called a *function of a function* or a *composite function*, defined in X. It is also said, that the function $\varphi[f(x)]$ is *a superposition of functions $f(x)$ and $\varphi(y)$*.

For example, if $y = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $z = e^{ay}$, $y \in [-1, 1]$, then $z = e^{a \sin x}$ is a composite function of x on the segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The rule of differentiating a composite function is expressed by means of the following theorem:

THEOREM 1. *If the function $y = f(x)$ has a derivative $y'(x) = f'(x)$ at the point $x \in X$, and the function $z = \varphi(y)$ has a derivative $z' = \varphi'(y)$ at the corresponding point $y \in Y$, then the composite function $z = \varphi[f(x)]$ also has a derivative z' at the point $x \in X$, which is equal to the product of the derivatives of functions $\varphi(y)$ and $f(x)$:*

$$z'_x = \{\varphi[f(x)]\}' = \varphi'_y(y) \,.\, f'_x(x) \equiv \varphi'_y[f(x)]f'_x(x)$$

or, for short

$$z'_x = z'_y \,.\, y'_x. \tag{1.6}$$

EXAMPLE 4. For the function $z = e^{\sin x}$, i.e. $y = \sin x$, $z = e^y$, we have

$$z'_x = e^y \cos x = e^{\sin x} \cos x.$$

Certain corollaries which can be obtained from this theorem play an important part in the technique of calculating derivatives.

COROLLARY 1. *The derivative of an inverse function.* Suppose there exists for the function $y = f(x)$ the inverse function $x = \varphi(y)$. Then, if the function $\varphi(y)$ has a derivative $\varphi'(y)$ other than zero, the function $f(x)$ has the derivative

$$f'(x) = \frac{1}{\varphi'(y)} = \frac{1}{\varphi'[f(x)]}.$$

This is written down in short

$$y'_x = \frac{1}{x'_y}, \tag{1.7}$$

where the suffixes of the derivatives show with respect to which variable the differentiation is taking place, i.e. which variable is considered as the independent one.

EXAMPLE 5. Let $y = \arcsin x$; then $x = \sin y$,

$$y'_x = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}.$$

The "+" sign should be ascribed to the root, since $-\frac{1}{2}\pi \leqslant \arcsin x \leqslant \frac{1}{2}\pi$. At points $y = \pm\frac{1}{2}\pi$, the derivative $x'_y = 0$ and at the points $x = \pm 1$ correspondingly, the derivative y'_x becomes $+\infty$.

COROLLARY 2. *The derivative of a function, given in parametric form.* If the functions $x(t)$ and $y(t)$ are defined and differentiable in

the set T and there exists a differentiable inverse function $t = t(x)$ defined in the set X, then the function $y = y(t)$ is a composite function of x:

$$y = y[t(x)],$$

defined in the set X. In this case it is said that the pair of functions

$$\left. \begin{array}{l} x = x(t), \\ y = y(t) \end{array} \right\} \tag{1.8}$$

define the function $y(x)$ *given in parametric form* in the set X. The function $y = y[t(x)]$ is differentiable in X and its derivative can be found by means of the formulae (1.6) and (1.7):

$$y'_x = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y'_t}{x'_t}. \tag{1.9}$$

COROLLARY 3. *Logarithmic derivative. The logarithmic derivative* of function $f(x)$ at the point $x = x_0 \in X$ is understood to be the derivative at point $x = x_0$ of the natural logarithm of the function $f(x)$ (or of the logarithm of its modulus when $f(x) < 0$) if this derivative exists. It is written usually

$$\left. \frac{d \ln f(x)}{dx} \right|_{x = x_0} \quad \text{or} \quad \left. \frac{d \ln |f(x)|}{dx} \right|_{x = x_0}.$$

From (1.6) we find (for $f(x) > 0$)

$$\left. \frac{d \ln f(x)}{dx} \right|_{x = x_0} = [\ln f(x)]' \Big|_{x = x_0} = \frac{f'(x_0)}{f(x_0)}. \tag{1.10}$$

In the same way (for $f(x) < 0$)

$$\left. \frac{d \ln |f(x)|}{dx} \right|_{x = x_0} = \ln |f(x)|]' \Big|_{x = x_0} = \frac{f'(x_0)}{f(x_0)}.$$

If, for example, t is the time during which a certain quantity $y = f(t)$ changes, then $f'(t)$ is the rate of change of the quantity $y = f(t)$ and $\dfrac{d \ln f(t)}{dt} = [\ln f(t)]$ is the *relative rate* of change of this quantity.

In a number of cases the logarithmic derivative can be used in finding the ordinary derivative.

Let us find, for example, the derivative of the function

$$f(x) = u_1(x)u_2(x)\dots u_n(x), \quad \text{where} \quad u_i(x) \neq 0 \quad (i = 1, 2, \dots, n).$$

Using the formula (1.10), we get

$$f'(x) = f(x)\frac{d \ln |f(x)|}{dx} = u_1(x)u_2(x)\dots u_n(x)\left(\sum_{k=1}^{n} \ln |u_k(x)|\right)'$$

$$= u_1(x)u_2(x)\dots u_n(x) \sum_{k=1}^{n} \frac{u_k'(x)}{u_k(x)}.$$

Formula (1.4) is a particular case of the formula obtained.

The formula (1.5) can be deduced in a similar manner.

A function of the form

$$f(x) = u(x)^{v(x)}, \quad \text{where} \quad u(x) > 0,$$

is called *a composite exponential function.* Its derivative is equal to

$$f'(x) = f(x)\,[\ln f(x)]' = u(x)^{v(x)}[v(x) \ln u(x)]'$$

$$= u(x)^{v(x)}\left[v'(x) \ln u(x) + v(x)\frac{u'(x)}{u(x)}\right].$$

Thus,

$$(u^v)' = u^v \ln u \,.\, v' + vu^{v-1}u'. \tag{1.11}$$

3. The name *elementary functions* is taken to mean a class of functions including power, rational (whole-number and fractional) exponential, logarithmic, trigonometric and inverse trigonometric functions, and also their combinations in the form of various superpositions. In order to define this class of functions rigorously, we can confine ourselves to the investigation of some of them, which we accept as principal ones.

Let us give the name *elementary operation* to any of the following operations:

(a) arithmetical operations;

(b) the operations of finding the values of functions e^x, $\ln x$ ($x > 0$), $\sin x$, $\arcsin x$ (when $|x| \leqslant 1$), given x.

We shall call the function $f(x)$ *elementary* if, for a given x belonging to the domain of definition of the function f, the value of $f(x)$ is determined by a finite sequence of elementary operations independent of x.

The following may be quoted as examples of elementary functions:

(1) all functions mentioned in (b);

(2) any polynomials $P_n(x)$ and rational functions $R(x)$;

(3) functions $\cos x = \sin(\frac{1}{2}\pi - x)$, $\tan x = \dfrac{\sin x}{\sin(\frac{1}{2}\pi - x)}$, $x^\alpha + e^{\alpha \ln x}$, $\arctan x = \arcsin \dfrac{x}{\sqrt{1+x^2}}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and others;

(4) more complicated combinations, something like

$$\arctan(e^x + \sin x^3), \text{ etc.}$$

Formulae (1.2)–(1.7) enable us to find an expression for the derivative of any elementary function, and this leads to the conviction, that *the derivative of an elementary function is itself always an elementary function.* Thus, *the operation of differentiation does not lead out of the class of elementary functions.*

Integration—an operation, inverse to the operation of differentiation—does not possess this property. For example, the functions discussed in section 4 of the appendices, $\mathrm{si}(x)$, $\mathrm{ci}(x)$, $\mathrm{Ei}(x)$, $\mathrm{li}(x)$, $\Phi(x)$, $S(x)$, $C(x)$ and others, obtained by integration of the elementary functions

$$\frac{\sin x}{x},\quad \frac{\cos x}{x},\quad \frac{e^x}{x},\quad \frac{1}{\ln x},\quad \sqrt{\frac{2}{\pi}}\, e^{-x^2/2},\quad \sqrt{\frac{2}{\pi}} \sin(x^2),\quad \sqrt{\frac{2}{\pi}} \cos(x^2)$$

and others, are not themselves elementary functions.

4. Below, a number of theorems are given, which enable us to say something about a derivative by the properties of the function, or vice versa. Of these, particularly important are *the theorem of mean value* in differential calculus. (Rolle, Lagrange, Cauchy theorems).

(a) Let the function $y = f(x)$ be defined in the set X and let it have a finite $y'_x = f'(x_0)$ at the point $x = x_0 \in X$.

Then the increment

$$\Delta y = \Delta f(x_0) = f(x_0 + h) - f(x_0) \qquad (x_0 + h \in X)$$

can be represented in the form

$$\Delta f(x_0) \equiv f(x_0 + h) - f(x_0) = f'(x_0)h + o(h)$$

or

$$\Delta y = y'_x h + o(h) \tag{1.12}$$

(about the symbol $o(h)$, see volume 69 of this series).

(b) If the function $f(x)$ has a finite derivative $f'(x)$ at the point $x = x_0 \in X$, then $f(x)$ is continuous at $x = x_0$.

(c) If the function $f(x)$ has a derivative in the vicinity of point $x = x_0$ and if it tends to infinity, when $x \to x_0$, then its derivative cannot remain bounded in this case.

(d) THEOREM 2 *(Rolle). Suppose the function $f(x)$ is defined and is continuous in the segment $[a, b]$, has a finite derivative at every point of the interval (a, b) and $f(a) = f(b)$, then there can be found a point $\xi \in (a, b)$ at which the derivative becomes zero:* $f'(\xi) = 0$.

(e) THEOREM 3 *(Lagrange). If the function $f(x)$ is defined and is continuous in the segment $[a, b]$ and has a finite derivative at every point of the interval (a, b), then there can be found a point $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(\xi),$$

or

$$f(b) - f(a) = f'(\xi)(b - a). \tag{1.13}$$

The formula (1.13) is called *Lagrange's formula*, or *the formula of finite increments*. In particular, for the segment $[x, x + h]$, lying inside $[a, b]$, the formula (1.13) is written down in the form

$$f(x + h) - f(x) = hf'(x + \vartheta h),$$

where $0 < \vartheta < 1$.

The geometrical sense of this theorem is as follows: on the curve $y = f(x)$ (Fig. 1) which has a tangent at every point, there can be found at least one point in the interval (a, b) at which the tangent to the curve $y = f(x)$ is parallel to the chord MN passing through the points $M(a, f(a))$, $N(b, f(b))$ (Fig. 1 shows two such points, K_1 and K_2).

A generalization of Lagrange's theorem is

(f) THEOREM 4 *(Cauchy). Suppose the functions $f(t)$ and $\varphi(t)$ are defined and are continuous in the segment $[a, b]$, have finite derivatives $f'(t)$ and $\varphi'(t)$ at every point of the interval (a, b)* and $\varphi'(t) \neq 0$. *Then there can be found a point $\xi \in (a, b)$, such that*

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(\xi)}{\varphi'(\xi)}, \tag{1.14}$$

or

$$[f(b) - f(a)]\, \varphi'(\xi) = [\varphi(b) - \varphi(a)] f'(\xi).$$

The geometrical sense of Cauchy's theorem is as follows: the curve (Fig. 1) is given in parametric form $x = \varphi(t)$, $y = f(t)$; the point M corresponds to the value of the parameter $t = a$ and has

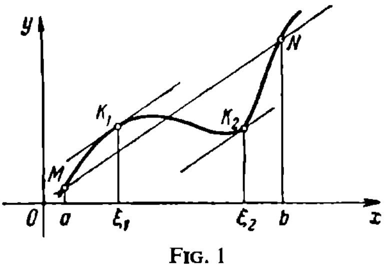

FIG. 1

coordinates $(\varphi(a), f(a))$ and the point M has coordinates $(\varphi(b), f(b))$. The gradient of the tangent to the curve at the point ξ equals

$$\tan \alpha = y'_x = \frac{f'(\xi)}{\varphi'(\xi)},$$

and the gradient of the chord is

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)}.$$

Thus, Cauchy's theorem confirms the equality of gradients, i.e. the parallelness of the tangent and the chord.

Lagrange's theorem can be obtained as a particular case of Cauchy's theorem if we put $t = x$ and $\varphi(x) = x$ in it.

Let us also note the properties of derivatives for even and odd functions.

(g) *The derivative of an even function is an odd function, conversely, the derivative of an odd function is even. The simplest examples are the equalities*

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x.$$

5. Suppose $f(x)$ is obtained as a result of applying elementary operations (in the sense of section 3) to the continuous functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$. Then, generally speaking, $\lim\limits_{x \to x_0} f(x)$ can be

obtained as a result of applying the same operations to the values of $\varphi_1(x_0), \varphi_2(x_0), \ldots, \varphi_n(x_0)$. For example,

$$\lim_{x \to \frac{\pi}{4}} \frac{e^x - \sin x}{x^2 + x + 1} = \frac{e^{\frac{\pi}{4}} - \sin \frac{\pi}{4}}{\left(\frac{\pi}{4}\right)^2 + \frac{\pi}{4} + 1}.$$

If these operations, or at least one of them, turn out to be impossible to carry out, it may happen, that $\lim\limits_{x \to x_0} f(x)$ is found on the basis of theorems about arithmetical operations on infinitely small or infinitely great quantities. For example,

$$\lim_{x \to +0} \frac{1}{\ln x} = 0,$$

because, although the function $\ln x$ is not defined when $x = 0$, it is known, that $\lim\limits_{x \to +0} \ln x = -\infty$, i.e. $\ln x$ is an infinitely great quantity, and a quantity, inverse of an infinitely great one, tends to zero.

However, for example,

$$\lim_{x \to 0} \frac{x - \sin x}{1 - \cos x}$$

and in general

$$\lim_{x \to x_0} \frac{\varphi(x)}{\psi(x)},$$

where

$$\varphi(x_0) = \psi(x_0) = 0,$$

cannot be found on the basis of these theorems. In this case it is said that an "indeterminacy" is obtained, or an indeterminate expression of the form 0/0.

In a similar sense, other indeterminate expressions of the form

$$\frac{\infty}{\infty}, \quad 0 \,.\, \infty, \quad \infty - \infty, \quad 1^{\infty}, \quad 0^0, \quad \infty^0$$

are used.

In many cases, the limits of functions that lead to indeterminate expressions can be found by means of elementary transformations

of the given function, for example

$$\lim_{x\to a} \frac{x^2 - a^2}{x^3 - a^3} = \lim_{x\to a} \frac{x + a}{x^2 + ax + a^2} = \frac{2}{3a}.$$

The general method of finding limits of indeterminate expressions is based on the application of one of the theorems quoted below. They are usually grouped under the general name *l'Hôpital's rule.* The first few of these theorems refer to the indeterminate expressions of the form 0/0.

THEOREM 5. *Suppose functions $f(x)$ and $\varphi(x)$ are defined in the vicinity of the point a and $f(a) = \varphi(a) = 0$. If there exist finite derivatives $f'(a)$ and $\varphi'(a)$, where $\varphi'(a) \neq 0$, then*

$$\lim_{x\to a} \frac{f(x)}{\varphi(x)} = \frac{f'(a)}{\varphi'(a)}.$$

If both derivatives become zero at the point $x = a$, Theorem 5 is inapplicable. Then, its generalizations should be consulted. These generalizations can be carried out in various directions. The first one requires an approach to derivatives of higher orders (see § 2).

THEOREM 6. *Suppose the functions $f(x)$ and $\varphi(x)$ are defined in the vicinity of the point a and become zero at point a together with their derivatives up to the order $n - 1$ inclusively, and derivatives $f^{(n)}(a)$ and $\varphi^{(n)}(a)$ exist and are finite, while $\varphi^{(n)}(a) \neq 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{\varphi(x)} = \frac{f^{(n)}(a)}{\varphi^{(n)}(a)}.$$

In both the cases considered, the existence of a limit is guaranteed by the fulfilment of the conditions of the corresponding theorems. The second direction of generalization of Theorem 5 is represented by the following theorem, in which the finding of a limit of the ratio of the functions is reduced to the finding of the limit of the ratios of their derivatives.

THEOREM 7. ***Suppose functions $f(x)$ and $\varphi(x)$ are defined in the semi-open interval $(a, b]$ and satisfy the following conditions:***

(1) $\lim\limits_{x\to a} f(x) = 0$, $\lim\limits_{x\to a} \varphi(x) = 0$;

(2) ***in the semi-open interval $(a, b]$ there exist finite derivatives $f'(x)$ and $\varphi'(x)$, where $\varphi'(x) \neq 0$.***

Then, if there exists a limit (finite or infinite) of the ratio of derivatives $\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$ *then a limit of the ratio of the functions does exist and is equal to*

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}. \tag{1.15}$$

This theorem represents wider possibilities, inasmuch as the limit of the ratio of derivatives can very often be obtained by elementary methods. It also gives us the possibility of a repeated application. Here, all kinds of simplifications of the expressions obtained are admissible, the division by common factors, the utilization of limits already known. The extension of the theorem to cover the case $a=\infty$ is also possible.

THEOREM 8. *Suppose the functions $f(x)$ and $\varphi(x)$ are defined in the semi-open interval $[c, +\infty)$ and satisfy the following conditions:*

(1) $\lim\limits_{x\to+\infty} f(x)=0, \quad \lim\limits_{x\to+\infty}\varphi(x)=0;$

(2) *in the semi-open interval $[c, +\infty)$ there exist finite derivatives $f'(x)$ and $\varphi'(x)\neq 0$.*

Then, if there exists a limit (finite or infinite) of the ratio of derivatives, $\lim\limits_{x\to+\infty}\frac{f'(x)}{\varphi'(x)}$, *a limit of the ratio of functions does exist and is equal to*

$$\lim_{x\to+\infty}\frac{f(x)}{\varphi(x)}=\lim_{x\to+\infty}\frac{f'(x)}{\varphi'(x)}.$$

It is necessary to keep in mind, that if no limit of the ratio of derivatives exists, Theorems 7 and 8 cannot be used, although a limit of the ratio of functions may still exist.

EXAMPLE 6. Find the limit

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{2\sin x}=\left.\frac{e^x+e^{-x}}{2\cos x}\right|_{x=0}=1.$$

Here we used Theorem 5.

EXAMPLE 7. Find the limit

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}.$$

Here $f(x)=e^x-e^{-x}-2x$, $f(0)=0$; further

$$f'(0)=(e^x+e^{-x}-2)|_{x=0}=0,$$
$$f''(0)=(e^x-e^{-x})|_{x=0}=0,$$
$$f'''(0)=(e^x+e^{-x})|_{x=0}=2.$$

In the same way

$$\varphi(x)=x-\sin x,\quad \varphi(0)=0,$$
$$\varphi'(0)=(1-\cos x)|_{x=0}=0,$$
$$\varphi''(0)=\sin x|_{x=0}=0,$$
$$\varphi'''(0)=\cos x|_{x=0}=1.$$

Thus, on applying Theorem 6 for $n = 3$, we find that the required limit is equal to 2.

EXAMPLE 8. Find the limit

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}.$$

Here, the application of Theorem 6 (for $n = 3$) is possible. However, it is shorter to apply Theorem 7:

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = \lim_{x\to 0} \frac{\dfrac{1}{\cos^2 x} - 1}{1 - \cos x} = \lim_{x\to 0} \left(\frac{1}{\cos^2 x} \cdot \frac{1 - \cos^2 x}{1 - \cos x}\right)$$

$$= \lim_{x\to 0} \frac{1 + \cos x}{\cos^2 x} = 2.$$

EXAMPLE 9. The limit of the ratio

$$\lim_{x\to 0} \frac{x^2 \sin \dfrac{1}{x}}{\sin x}$$

cannot be found in accordance with l'Hôpital's rule, although it does represent an example of an indeterminacy of the form 0/0. The ratio of derivatives has the form

$$\frac{2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}}{\cos x}$$

and, for $x \to 0$, it does not tend to any limit. At the same time the limit of the original expression does exist and equals zero, which can be verified easily by representing this expression in the form:

$$\lim_{x\to 0} \frac{x^2 \sin \dfrac{1}{x}}{\sin x} = \lim_{x\to 0} \left\{x \frac{x}{\sin x} \sin \frac{1}{x}\right\} = 0.$$

Indeed, the first of these factors tends to zero, the other one—to one, and the third one remains bounded.

EXAMPLE 10. It may happen, that the limit of the ratio of derivatives exists, while the limit of the ratio of functions does not. For example, when $x \to +\infty$, the ratio

$$\frac{e^{-2x} (\cos x + 2 \sin x)}{e^{-x} (\cos x + \sin x)}$$

has the form 0/0. The ratio of the derivatives equals

$$\frac{-5e^{-2x} \sin x}{-2e^{-x} \sin x} = \frac{5}{2} e^{-x}$$

and for $x \to +\infty$ it has a limit equal 0. Having now represented the ratio of the functions in the form

$$\frac{e^{-2x}(\cos x + 2\sin x)}{e^{-x}(\cos x + \sin x)} = e^{-x}\left(1 + \frac{1}{1 + \cot x}\right),$$

it is easy to see that this ratio does not tend to any limit for $x \to +\infty$, since the factor $[1 + 1/(1 + \cot x)]$ oscillates all the time between $-\infty$ and $+\infty$.

Theorem 8 is inapplicable here, since the derivative $\varphi'(x)$ contains the factor $\sin x$, which becomes zero in any semi-open interval $[c, +\infty)$ and the condition (2) of the Theorem 8 is not fulfilled.

The following theorems refer to indeterminate expressions of form ∞/∞.

THEOREM 9. *Suppose the functions $f(x)$ and $\varphi(x)$ are defined in the semi-open interval $(a, b]$ and satisfy the conditions:*

(1) $\lim\limits_{x\to a} f(x) = \infty, \quad \lim\limits_{x\to a} \varphi(x) = \infty;$

(2) *in the semi-open interval $(a, b]$ there exist finite derivations $f'(x)$ and $\varphi'(x)$, and $\varphi'(x) \neq 0$.*

Then, if there exists a limit (finite or infinite) of the ratio of derivatives, $\lim\limits_{x\to a} \dfrac{f'(x)}{\varphi'(x)}$, there also exists a limit of the ratio of the functions, and it equals

$$\lim_{x\to a} \frac{f(x)}{\varphi(x)} = \lim_{x\to a} \frac{f'(x)}{\varphi'(x)}.$$

THEOREM 10. *Suppose the functions $f(x)$ and $\varphi(x)$ are defined in the semi-open interval $[c, +\infty)$ and satisfy the conditions:*

(1) $\lim\limits_{x\to+\infty} f(x) = \infty, \quad \lim\limits_{x\to+\infty} \varphi(x) = \infty;$

(2) *in the semi-open interval $[c, +\infty)$ there exist finite derivatives $f'(x)$ and $\varphi'(x) \neq 0$.*

Then, if there exists a limit (finite or infinite) of the ratio of derivatives, $\lim\limits_{x\to+\infty} \dfrac{f'(x)}{\varphi'(x)}$, a limit of the ratio of the functions exists and is equal to

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi(x)} = \lim_{x\to+\infty} \frac{f'(x)}{\varphi'(x)}.$$

Indeterminate expressions of other forms are usually transformed beforehand into the form 0/0 or ∞/∞, which can be done by means of elementary transformations. For example, the product $f(x)\,\varphi(x)$, where $f(x) \to 0$ and $\varphi(x) \to \infty$, representing an indeterminacy of the form $0 \times \infty$ can be written in the form

$$\frac{f(x)}{1/\varphi(x)} \quad \text{or} \quad \frac{\varphi(x)}{1/f(x)}.$$

Here the first expression represents an indeterminacy of the form 0/0, and the second one—of the form ∞/∞.

Similarly, the expression $f(x) - \varphi(x)$, in which $f(x) \to \infty$, $\varphi(x) \to \infty$ (indeterminacy of the type $\infty - \infty$), can be reduced to the form

$$\left(\frac{1}{f(x)} - \frac{1}{\varphi(x)}\right) : \frac{1}{f(x)\,\varphi(x)},$$

representing an indeterminacy of the form $0/0$.

As to indeterminate expressions of form 1^∞, 0^0, ∞^0, it is advisable to take their logarithms, when they assume one of the forms discussed earlier.

6. As was shown in section 4, a function, differentiable in the set X, is continuous in it. The converse conclusion is not true: there exist functions, continuous all over E_1, and not possessing a derivative at any point. The first of such examples was constructed by B. Bolzano. We now give examples of such functions.

Weierstrass' function is defined by the series

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, and b is an odd natural number, which is selected in such a way, that $ab > 1 + \frac{3}{2}\pi$. The series defining Weierstrass' function has as a majorant the series $\sum\limits_{n=0}^{\infty} a^n$ and therefore it converges uniformly in E_1, as a result of which $f(x)$ is continuous everywhere. However, the function $f(x)$ does not have a finite derivative at any point. At the same time, $f(x)$ does have an infinite derivative at every point.

Van der Waerden's function is defined with the help of the function $u_0(x)$. We put $u_0(x) = |s - x|$, where s is the integer nearest in value to x. The function $u_0(x)$ is continuous; its graph is a broken line, the gradients of the links being equal to ± 1.

We define the functions $u_k(x)$, when $k = 1, 2, \ldots$, by means of equations

$$u_k(x) = \frac{u_0(4^k x)}{4^k}.$$

Then $u_k(x)$ is linear in the segments $\left[\frac{s}{2\,.\,4^k}, \frac{s+1}{2\,.\,4^k}\right]$. Van der Waerden's function

$$f(x) = \sum_{k=0}^{\infty} u_k(x)$$

is continuous everywhere, and does not possess either a finite or an infinite derivative at any point.

7. Suppose the function $f(x)$ is continuous in the set X and has a *finite* derivative $f'(x)$ at every point of it. This derivative may be a continuous as well as a discontinuous function.

Let us consider, for example, the function

$$\varphi(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

This function is continuous for all values of x. If $x \neq 0$, then

$$\varphi'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

For $x = 0$

$$\varphi'(0) = \lim_{h \to 0} \frac{h^2 \sin \dfrac{1}{h}}{h} = 0.$$

Thus $\varphi'(x)$ exists for all values of x.

But, since, when $x \to 0$, the function $2x \sin (1/x)$ tends to zero, and $\cos (1/x)$ oscillates between -1 and $+1$, $\varphi'(x)$ has no limit, when $x \to 0$ and, therefore, $\varphi'(x)$ is discontinuous at the point $x = 0$. Here, $x = 0$ is a discontinuity of the second kind. If, as was supposed, the function $f(x)$ is differentiable at all points of the set X, then its derivative cannot have discontinuities of the first kind; all its discontinuities, if they exist, are those of the second kind.

A derivative, even when discontinuous, possesses a property belonging to continuous functions, namely, the following theorem holds:

THEOREM 11 *(Darboux). If the function $\varphi(x)$ has a definite derivative $\varphi'(x)$ at every point of the interval $[a, b]$, then $\varphi'(x)$ takes in the interval $[a, b]$ all values included between $\varphi'(a)$ and $\varphi'(b)$.*

In particular, it follows hence, that if the derivative has different signs at the points a and b, then inside the interval $[a, b]$ there is at least one point ξ, in which $\varphi'(\xi) = 0$.

8. The differential of a function of one variable can be defined in two different ways.

(a) *The differential as the principal linear part of the increment.* For the function $y = f(x)$, defined in the set X and continuous at the point $x = x_0 \in X$, the increment

$$\Delta y = \Delta f(x_0) = f(x_0 + h) - f(x_0), \quad x_0 + h \in X,$$

is an infinitely small quantity together with h. If the following representation takes place

$$\Delta y = Ah + o(h), \tag{1.16}$$

where A is constant relative to h, the expression Ah is called the *differential* of the function $f(x)$ at the point x_0 and is denoted

$$dy \equiv df(x_0) \equiv df(x_0, h).$$

Thus,

$$df(x_0, h) = Ah. \tag{1.17}$$

In order that it should be possible to represent the increment of a function in the form (1.16), it is necessary and sufficient that a finite derivative $f'(x)$ of the function $f(x)$ exists at the point x_0, i.e. that the function is *differentiable*. This representation is *unique* i.e. it takes place for one possible value of A only. It turns out, that this value is $A = f'(x_0)$ and the equation (1.16) is rewritten in the form

$$\Delta y = \Delta f(x_0) = f'(x_0)h + o(h) \tag{1.18}$$

and

$$df(x_0, h) = f'(x_0)h. \tag{1.19}$$

Assuming the differential of the independent variable to be *by definition* equal to its increment, it is possible to make use of the notation $dx = h$. Then, in place of (1.19), we write

$$dy = f'(x_0)\, dx. \tag{1.20}$$

The expression $f'(x_0)\, dx$ is *the principal linear part* of the increment $\Delta f(x)$, since it is a linear function with respect to dx and coincides with the increment $\Delta f(x)$ to the first order in h.

This is used in approximate calculations, when the increment Δy is replaced by the differential $f'(x_0)\, h$; here, the error admitted is $\alpha = 0(h)$.

EXAMPLE 11.

$$\sin 46° = \sin\left(\frac{\pi}{4} + \frac{\pi}{180}\right) \approx \sin\frac{\pi}{4} + \cos\frac{\pi}{4}\cdot\frac{\pi}{180} = 0{\cdot}7194,$$

whereas the true value of $\sin 46° = 0{\cdot}7193\ldots$

(b) *The differential as a derivative of the parameter.* Suppose the function $f(x)$ is defined in the vicinity of the point $x = x_0$, and t and h are real numbers (it is assumed that the point $x = x_0 + th$ belongs to the set X when $t \in [0, 1]$). Then, for fixed x_0 and h the function $f(x) = f(x_0 + th)$ is a function of t. If, for the function $f(x)$, there exists a derivative $f'(x_0)$, then there also exists

$$\frac{d}{dt} f(x_0 + th)|_{t=0} = f'(x_0)h = df(x_0, h) \equiv df(x_0). \tag{1.21}$$

If the parameter $t \in [0, 1]$ is interpreted as time, the point $x_t = x_0 + th$ moves uniformly along the axis Ox from the point x_0 to the point $x_0 + h$ with velocity $|h|$.

When the point x_t moves in this way, the left-hand side of the equation (1.20) means the rate of change of function $f(x_t)$ at the initial instant $t = 0$.

Thus, the differential $df(x_0, h)$ can be defined as the derivative with respect to a parameter $\frac{d}{dt} f(x_0 + th)|_{t=0}$. This definition can be easily generalized to cover functions of several variables (see Chapter II, § 1) and also functions of a more general nature. Let us now consider the differential of a composite function $y = f[\varphi(t)]$, which is a superposition of the functions $x = \varphi(t)$ and $y = f(x)$. If there exist derivatives $y'_x = f'(x)$ and $x'_t = \varphi'(t)$, then, on the basis of the theorem in section 2, there also exists a derivative $y'_t = y'_x x'_t$ (see (1.6)).

If we regard x as an independent variable, the differential dy of function $y = f(x)$ is expressible according to the formula (1.20):

$$dy = y'_x \, dx = f'(x) \, dx.$$

If we regard t as the independent variable, and y and x as functions of t, then

$$dy = y'_t \, dt = y'_x \, dx, \quad \text{where} \quad dx = x'_t \, dt.$$

Thus, the form of expression of the differential

$$dy = f'(x) \, dx \tag{1.20'}$$

does not change when the independent variable x is exchanged for t; it is the sense of the notation dx that changes: in the formula (1.20) dx means the arbitrary increment h of the variable x, and in the formula (1.20′) it means the differential of function $x = \varphi(t)$ of the variable t. This property is called the *invariance of the form of the first differential.*

The simplest generalizations of the concept of derivative

9. Suppose the function $y = f(x)$ is defined in the set X, $x_0 \in X$ and

$$\Delta y \equiv \Delta f(x_0) \equiv f(x_0 + h) - f(x_0)$$

is the increment of the function $f(x)$ at the point $x = x_0$ $(x_0 + h \in X)$. For $h \to 0$, the ratio $[\Delta f(x_0)]/h$ may not necessarily have a definite limit. If, for this

ratio, there exists a limit from the left (see volume 69 of this series) then it is called ***a left-hand derivative*** or ***a derivative from the left*** of the function $f(x)$ at the point $x = x_0$ and it is denoted $f'_-(x_0)$:

$$f'_-(x_0) = \lim_{h\to -0} \frac{f(x_0+h)-f(x_0)}{h}. \tag{1.22}$$

Similarly, the limit from the right of the ratio $[\Delta f(x_0)]/h$, if it exists, is called ***the derivative from the right*** or ***the right-hand derivative*** of the function $f(x)$ at the point $x = x_0$:

$$f'_+(x_0) = \lim_{h\to +0} \frac{f(x_0+h)-f(x_0)}{h}. \tag{1.23}$$

The left-hand derivative and the right-hand derivative of the function $f(x)$ are called ***one-sided derivatives.*** If the corresponding one-sided limit equals infinity of a definite sign, the function has at that point ***an infinite one-sided derivative.***

For example, the function

$$f(x)\begin{cases} 0 & \text{for} \quad x \leqslant 1, \\ x-1 & \text{for} \quad x > 1 \end{cases}$$

has, at the point $x_0 = 1$, a left-hand and a right-hand derivative equal, respectively, to

$$f'_-(1) = 0, \quad f'_+(1) = +1.$$

The function $f(x) = |x| \equiv x \operatorname{sign} x$, defined for all $x \in E$, has both one-sided derivatives at the point $x_0 = 0$, and

$$f'_-(0) = -1, \quad f'_+(0) = +1.$$

The function $y = x^{2/3}$ has, for $x_0 = 0$, the one-sided derivatives

$$f'_-(0) = -\infty, \quad f'_+(0) = +\infty.$$

We now note some propositions about one-sided derivatives:

(a) If the function $f(x)$ is continuous at the point $x = x_0 \in X$ and has equal one-sided derivatives (finite or infinite) $f'_-(x_0) = f'_+(x_0) = a$, then $f(x)$ has a derivative $f'(x_0)$ at the point $x = x_0$ and $f'(x_0) = a$.

(b) If the function $f(x)$ is continuous at points a and b of X, is differentiable for all $x \in (a, b)$ and there exist a right-hand and a left-hand limit $f'(a+0)$ and $f'(b-0)$, then the function $f(x)$ has one-sided derivatives $f'_+(a)$ and $f'_-(b)$, and

$$f'_+(a) = f'(a+0), \quad f'_-(b) = f'(b-0).$$

These equalities can be regarded as a definition of one-sided derivatives. It is more convenient for generalizations (see § 2) but narrower, than the original one.

If both one-sided derivatives of the function $y = f(x)$ at the point $x = x_0 \in X$ exist and are different, $f'_-(x_0) \neq f'_+(x_0)$, such a point is called a node. If the derivatives on the right and on the left both equal infinity of opposite signs, the point is called ***a cusp*** (more precisely, ***a cusp of the 1st kind***).

For example, the point $x = 0$ for the function $y = |x|$ is an angular point (Fig. 2), and for the function $y = x^{2/3}$, it is a cuspidal point of the 1st kind (Fig. 3).

Another generalization of the concept of a derivative is *the symmetrical derivative (Schwartzian derivative)*. The name of symmetrical derivative $f^{[']}(x)$ of the function $f(x)$ at the point $x = x_0 \in X$ is given to the limit of the ratio

$$f^{[']}(x) = \lim_{h\to 0} \frac{\bar{\Delta}_h f(x_0)}{2h} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h},$$

if this limit exists and is finite.

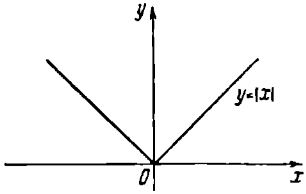

FIG. 2

If a function $f(x)$ possesses an ordinary derivative $f'(x_0)$ at the point $x = x_0$, then the symmetrical derivative $f^{[']}(x)$ also exists at that point and it coincides with the ordinary one

$$f^{[']}(x_0) = f'(x_0).$$

The converse is *not* true: a symmetrical derivative may exist at a given point even without the existence of an ordinary one. For example, in the case of the function

$$y = \begin{cases} -x & \text{when } x < 0, \\ 2x & \text{when } x \geqslant 0 \end{cases}$$

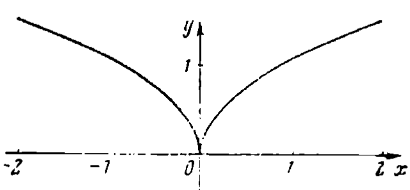

FIG. 3

there is no ordinary derivative when $x_0 = 0$, but the Schwartzian derivative of this function at the point $x_0 = 0$ does exist and equals $\frac{1}{2}$. The function $y = |x|$ at $x_0 = 0$ possesses a symmetrical derivative, which equals zero, and it also does not possess an ordinary derivative there.

In general, if $f(x)$ is any function, it always possesses at the point $x_0 = 0$ a

Schwartzian derivative which equals zero, even in the case when $f(x)$ is discontinuous at that point. For example, for the function

$$f(x) = \begin{cases} \dfrac{\cos x}{x^2} & \text{when} \quad x \neq 0, \\ 0 & \text{when} \quad x = 0 \end{cases}$$

the symmetrical derivative is

$$f^{[']}(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0 - h)}{2h} = \lim_{h \to 0} \frac{\dfrac{\cos h}{h^2} - \dfrac{\cos h}{h^2}}{2h} = 0.$$

§ 2. Derivatives and Differentials of Higher Orders. Taylor's Series

1. Suppose the function $f(x)$ is defined in the neighbourhood (a, b) of the point $x = x_0 \in (a, b)$ and we have the first derivative $f'(x)$ there. Then the name of *second successive derivative*, or, simply, *second derivative*, $f''(x_0) = \left.\dfrac{d^2y}{dx^2}\right|_{x=x_0}$ of the function $f(x)$ at the point $x = x_0$ is given to the derivative at the point $x = x_0$ of the derivative $f'(x)$, if the latter is differentiable:

$$f''(x_0) = [f'(x)]'|_{x = x_0}.$$

For example, when $x \neq 0$, we can write $(\ln x)'' = 1/x = -1/x^2$.

The second difference $\Delta_h^2 f(x_0)$ of function $f(x)$ at the point $x = x_0$ is the name given to the increment $\Delta_h u(x_0)$, where $u(x_0) = \Delta_h f(x_0)$:

$$\Delta_h^2 f(x_0) = \Delta_h(\Delta_h f(x_0)) = f(x_0 + 2h) - 2f(x_0 + h) + f(x_0).$$

Here, the function $f(x)$ is assumed to be defined in the set X, and $x_0 + kh \in X$ for $k = 0, 1, 2$.

The second differential derivative $\tilde{f}''(x_0)$ of the function $f(x)$ at the point $x = x_0 \in X$ is the name given to the limit of the ratio $[\Delta_h^2 f(x_0)]/h^2$, when $h \to 0$, if such a limit exists:

$$\tilde{f}''(x_0) = \lim_{h \to 0} \frac{\Delta_h^2 f(x_0)}{h^2} \equiv \lim_{h \to 0} \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2}. \tag{1.24}$$

If the function $f(x)$ possesses in the neighbourhood of point $x \in X$ a continuous second successive derivative $f''(x)$, then it also possesses

in the same neighbourhood a second differential derivative $\tilde{f}''(x)$ and these derivatives coincide. Conversely, if in the neighbourhood (a, b) of the point $x = x_0$ there exists a *continuous* second differential derivative $\tilde{f}''(x_0)$ to which the ratio $[\Delta_h^2 f(x)]/h^2$ tends *in a uniform manner* within (a, b) then, in the interval (a, b), there exists for the function $f(x)$ a second successive derivative $f''(x)$ and $f''(x) = \tilde{f}''(x)$, i.e. both definitions of the second derivative coincide.

If we give up the additional assumptions indicated above, then from the existence of a second derivative in the sense of one definition it does not follow that a second derivative exists in the sense of the second definition. For example, for the function

$$f(x) = \begin{cases} x^3 \sin \dfrac{1}{x} & \text{when} \quad x \neq 0, \\ 0 & \text{when} \quad x = 0 \end{cases}$$

for any $x \in E$, there exists the first derivative

$$f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x},$$

and $f'(0) = 0$. At the point $x_0 = 0$ the function $f(x)$ has a second differential derivative, equal zero, since

$$\frac{\Delta_h^2 f(0)}{h^2} = \frac{8h^3 \sin \dfrac{1}{2h} - 2h^3 \sin \dfrac{1}{h}}{h^2} \to 0$$

when $h \to 0$. At the same time the second successive derivative does not exist, because the expression

$$\frac{\Delta_h f'(0)}{h} = \frac{f'(0+h) - f'(0)}{h} = 3h \sin \frac{1}{h} - \cos \frac{1}{h}$$

has no limit when $h \to 0$.

2. By analogy with the above, *the n-th successive derivative*, or simply *the n-th derivative* (or, *the derivative of the n-th order*)

$$f^{(n)}(x_0) \equiv \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0}$$

of a function $f(x)$ at the point $x = x_0 \in X$ is the name given to the derivative at that point of the $(n-1)$th derivative $f^{(n-1)}(x)$, if $f^{(n-1)}(x)$ exists in some neighbourhood of the point x and is differ-

entiable there. In other words, the nth derivative is defined in a recurrent manner, by means of the derivative of the $(n-1)$th order

$$f^{(n)}(x_0) = [f^{(n-1)}(x)]'|_{x=x_0}. \qquad (1.25)$$

For example, $(\sin x)''' = (\cos x)'' = (-\sin x)' = -\cos x$, for $x \in (-\infty, +\infty)$.

If the function $f(x)$ has a derivative $f^{(n)}(x_0)$ at the point $x = x_0$, then it is continuous in the neighbourhood of point $x = x_0 \in X$ and has derivatives $f'(x), f''(x), \ldots, f^{(n-2)}(x)$, continuous in the neighbourhood of point $x = x_0$ and a derivative of $f^{(n-1)}(x)$ continuous *at the point* $x = x_0$.

In the same recurrent fashion we define, at the point $x = x_0 \in X$, *the n-th difference (the n-th increment)* $\Delta_h^n f(x_0)$ of the function $f(x)$ as the increment $\Delta_h u(x_0)$, where $u(x_0) = \Delta_h^{n-1} f(x_0)$ is the $(n-1)$th difference of function $f(x)$ at the point $x = x_0 \in X$:

$$\Delta_h^n f(x_0) = \Delta_h(\Delta_h^{n-1} f(x_0)) = \sum_{k=0}^{n} (-1)^k C_n^k f(x_0 + kh); \qquad (1.26)$$

here $f(x)$ is defined in X, $x_0 + kh \in X$ for all $k = 0, 1, \ldots, n$, and C_n^k denote binomial coefficients.

By definition we give the name *n-th differential derivative* $\tilde{f}^{(n)}(x_0)$ of function $f(x)$ at point $x = x_0 \in X$, to the limit of the ratio $[\Delta_h^n f(x_0)]/h^n$ when $h \to 0$, if such a limit exists.

Generally speaking, it does not follow from the existence of the differential derivative $\tilde{f}^{(n)}(x_0)$ of function $f(x)$ at some point $x = x_0 \in X$, that the nth successive derivative also exists. The converse holds when a supplementary assumption is made, as shown in the following theorem.

THEOREM 12. *If the function $f(x)$ possesses an n-th successive derivative $f^{(n)}(x_0)$ in a neighbourhood of the point $x = x_0 \in X$, which is continuous at the point $x = x_0$, then $f(x)$ possesses also an n-th differential derivative at that point.*

If there exists in the set X a finite nth successive derivative $f^{(n)}(x)$ of the function $f(x)$, it is said, that this function is *differentiable n times in X*. If, in addition, $f^{(n)}(x)$ is continuous in X, then $f(x)$ is called *continuously differentiable n times in X*. As was indicated above, in this case the two definitions of the nth derivative are equivalent, and $f^{(n)}(x)$ also coincides with the nth differential derivative.

3. Suppose the functions $u(x)$ and $v(x)$ have n continuous derivatives in the set X. The nth derivative of their product is given by *Leibniz's formula:*

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)} = uv^{(n)} + \frac{n}{1} u' v^{(n-1)} + \cdots$$
$$+ \frac{n(n-1)\dots(n-k+1)}{k!} u^{(k)} v^{(n-k)} + \cdots + u^{(n)} v. \tag{1.27}$$

This formula can be obtained by exchanging, in the formula of expansion of the binomial $(u + v)^n$, each index of power n_i for the symbol of differentiation of the n_ith order (n_i), if we agree, additionally, that $u^{(0)} = u$ and $v^{(0)} = v$.

The formula for the nth power of a sum is of the form

$$(u_1 + u_2 + \cdots + u_s)^n = \sum_{\substack{n_1, n_2, \dots, n_s \geqslant 0 \\ n_1 + n_2 + \cdots + n_s = n}} \frac{n!}{n_1! n_2! \dots n_s!} \times$$
$$\times\, u_1^{n_1} u_2^{n_2} \dots u_s^{n_s}. \tag{1.28}$$

If the functions $u_1(x), u_2(x), \dots, u_s(x)$ are differentiable continuously n times in the set X, a generalization of Leibniz's formula holds good:

$$(u_1 u_2 \dots u_s)^{(n)} = \sum_{\substack{n_1, n_2, \dots, n_s \geqslant 0, \\ n_1 + n_2 + \dots + n_s = n}} \frac{n!}{n_1! n_2! \dots n_s!} u_1^{(n_1)} u_2^{(n_2)} \dots u_s^{(n_s)}. \tag{1.29}$$

The right-hand side of the formula (1.29) is obtained from the right-hand side of formula (1.28) by exchanging the index of power n_i for the symbol of differentiation of the n_ith order (n_i).

4. The formulae for finding derivatives of higher orders for the composite function $F(x) = f[\varphi(x)]$ from the derivatives of the functions $f(x)$ and $\varphi(x)$ can be obtained by successive differentiations of the equation

$$F'(x) = f'(y)\, \varphi'(x),$$

which expresses the rule of differentiation of the composite function:

$$F''(x) = f''(y)\, [\varphi'(x)]^2 + f'(y)\, \varphi''(x),$$
$$F'''(x) = f'''(y)\, [\varphi'(x)]^3 + 3f''(y)\, \varphi'(x)\, \varphi''(x) + f'(y) \varphi'''(x),$$
$$F^{\mathrm{IV}}(x) = f^{\mathrm{IV}}(y)\, [\varphi'(x)]^4 + 6f'''(y)\, [\varphi'(x)]^2\, \varphi''(x)$$
$$+ f''(y)\, \{3[\varphi''(x)]^2 + 4\varphi'(x)\, \varphi'''(x)\} + f'(y)\, \varphi^{\mathrm{IV}}(x),$$

etc. The existence is assumed of all the derivatives shown.

5. Suppose, that the function $y = f(x)$ is defined and is differentiable in some neighbourhood (α, β) of the point x_0. The differential $df(x, h)$ is, for a fixed h, some function $\varphi(x)$ of the variable x in this neighbourhood:

$$\varphi(x) = df(x, h) = f'(x)h.$$

If there exists, at point $x = x_0$ a differential $d\varphi(x, h)$ of the function $\varphi(x)$ (of the differential of function $f(x)$), it is called the *second differential* of function $f(x)$ at the point $x = x_0$. The second differential of the function $f(x)$ is denoted by one of the symbols $d^2y = d^2f(x_0) = d^2f(x_0, h)$. By definition

$$d^2f(x_0, h) \equiv d\varphi(x_0, h) = d[df(x, h)]|_{x=x_0} = d[f'(x)h]|_{x=x_0}.$$

If the function $f(x)$ has a second differential at the point $x = x_0$, then there also exists a second derivative $f''(x_0)$ of the function $f(x)$ at that point; the converse proposition is also true; here

$$d^2y = d^2f(x_0, h) = f''(x_0)h^2 = f''(x_0)\,dx^2,$$

$$f''(x_0) = \left.\frac{d^2y}{dx^2}\right|_{x = x_0} = \left.\frac{d^2f(x)}{dx^2}\right|_{x = x_0}.$$

The differential of the n-th order (nth differential) $d^ny = d^nf(x_0) = d^nf(x_0, h)$ of the function $f(x)$ at the point $x = x_0$ is defined in a recurrent manner as the differential of its $(n-1)$th differential $d^{n-1}f(x_0, h)$, if the differential $d^{n-1}f(x_0, h)$ is defined in some neighbourhood of the point $x = x_0$:

$$d^ny \equiv d^nf(x_0) \equiv d^nf(x_0, h) = d[d^{n-1}f(x, h)]|_{x=x_0}.$$

If a function $f(x)$ has an nth differential $d^nf(x_0)$ at the point $x = x_0 \in X$, it also has an nth derivative $f^{(n)}(x_0)$ at this point and

$$d^nf(x_0, h) = f^{(n)}(x_0)h^n = f^{(n)}(x_0)\,dx^n. \tag{1.30}$$

The function $f(x)$, possessing an nth derivative $f^{(n)}(x_0)$ at the point $x = x_0 \in X$, also has an nth differential $d^nf(x_0, h)$, and

$$f^{(n)}(x_0)h^n = f^{(n)}(x_0)\,dx^n = d^nf(x_0, h).$$

The differential of the nth order can be defined as the nth derivative with respect to a parameter, analogously to the way in which

this was done in § 1, sec. 7. The following formula holds

$$d^n f(x_0, h) = \frac{d^n}{dt^n} [f(x_0 + th)]|_{t=0}, \tag{1.31}$$

which generalizes the formula (1.19) of the preceding section.

6. Suppose the function $f(x)$ is defined in the set X and in the intervals $(x_0, x_0 + h) \subset X$, $(x_0 - h, x_0) \subset X$ it has k derivatives. If, at the point $x = x_0$ there exists a limit to the right of the derivative $f^{(s)}(x)$, $s \leqslant k$, it is called the *s-th right-hand side derivative* $f_+^{(s)}(x_0)$ of the function $f(x)$ at the poin t$x = x_0 \in X$:

$$f_+^{(s)}(x_0) = f^{(s)}(x_0 + 0) = \lim_{x \to x_0 + 0} f^{(s)}(x). \tag{1.32}$$

Similarly, the limit to the left of the derivative $f^{(s)}(x) s \leqslant k$, at the point $x = x_0$, if it exists, is called the *s-th left-hand side derivative* $f^{(s)}(x_0)$ of the function $f(x)$ at the point $x = x_0 \in X$:

$$f_-^{(s)}(x_0) = f^{(s)}(x_0 - 0) = \lim_{x \to x_0 - x} f^{(s)}(x).$$

Left-hand and right-hand derivatives of the function $f(x)$ at the point $x = x_0$ are called its *one-sided derivatives.*

If a function $f(x)$ has, at the point $x = x_0$, k derivatives $f'(x_0), f''(x_0), \ldots, f^{(k)}(x_0)$, then, at that point, there exist all left- and right-hand derivatives of $f(x)$ up to the kth order inclusively, and

$$\begin{gathered} f'_-(x_0) = f'_+(x_0) = f'(x_0), \\ f''_-(x_0) = f''_+(x_0) = f''(x_0), \\ \cdots\cdots\cdots\cdots\cdots \\ f^k_-(x_0) = f^{(k)}_+(x_0) = f^{(k)}(x_0). \end{gathered}$$

If the function $f(x)$ has, at the point $x = x_0$, equal left- and right-hand derivatives up to the kth order inclusively, then it also has ordinary derivatives $f^{(i)}(x_0)$ $(i = 1, 2, \ldots, k)$ and

$$f^{(i)}(x_0) = f_-^{(i)}(x_0) = f_+^{(i)}(x_0) \qquad (i = 1, 2, \ldots, k).$$

At the given point, left- and right-hand derivatives may not coincide. For example, for the function $f(x) = e^{-|x|}$, $x \in E_1$, at the point $x_0 = 0$, there exist left- and right-hand derivatives of any order, and

$$f_+^{(k)}(0) = (-1)^k, \quad f_-^{(k)}(0) = +1 \qquad (k = 1, 2, \ldots).$$

This function does not possess an ordinary derivative at the point $x_0 = 0$. If the function $f(x)$ is defined in the interval $[a, b]$, then its derivatives at points a and b are understood to be one-sided derivatives, left-hand ones for $x = b$ and right-hand ones for $x = a$.

The second Schwartzian derivative (or the second symmetrical derivative) $f^{('')}(x_0)$ of the function $f(x)$ at the point $x = x_0 \in X$ can be defined in two

ways: *consecutively*, as a symmetrical derivative at the point $x = x_0$ of the first symmetrical derivative of the function $f(x)$:

$$f^{('')}(x_0) = [f^{(')}(x)]^{(')}|_{x=x_0}, \tag{1.33}$$

if $f^{(')}(x)$ exists in some vicinity of point $x = x_0$, and *by means of the second difference:*

$$\bar{\Delta}_h^2 f(x_0) \equiv \bar{\Delta}_h[\bar{\Delta}_h f(x_0)] = f(x_0 + 2h) - 2f(x_0) + f(x_0 - 2h), \tag{1.34}$$

as the limit of the ratio $[\Delta_h^{-2} f(x_0)]/4h$, when $h \to 0$, if this limit exists and is finite:

$$f^{('')}(x_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1) - 2f(x_0) + f(x_0 - h_1)}{h_1^2}, \tag{1.35}$$

where h is put equal to $2h$.

Similarly, ***the n-th Schwartzian derivative (the n-th symmetrical derivative)*** of the function $f(x)$ at the point $x = x_0$, can also be defined consecutively and by means of the nth difference. In the first case, it is defined as the Schwartzian derivative, at the point $x = x_0$, of the $(n-1)$th symmetrical derivative of function $f(x)$:

$$f^{(n)}(x_0) = [f^{(n-1)}(x)]^{(')}|_{x=x_0}, \tag{1.36}$$

if $f^{(n-1)}(x)$ is defined in some vicinity of the point $x = x_0 \in X$. In the second case, the nth difference of the function $f(x)$ is defined beforehand, by means of the equation

$$\bar{\Delta}_h^n f(x_0) = \bar{\Delta}_h[\bar{\Delta}_h^{n-1} f(x_0) = \sum_{k=0}^{n} (-1)^k C_n^k f\left[x_0 + \left(k - \frac{n}{2}\right)h\right], \tag{1.37}$$

which enables us to define the nth symmetrical derivative as the limit of the ratio $[\Delta_h^n f(x_0)]/h$ when $h \to 0$, if this limit exists and is finite:

$$f^{(n)}(x_0) = \lim_{h \to 0} \frac{\Delta_h^n f(x_0)}{h^n}. \tag{1.38}$$

7. Suppose $P_n(x)$ is a polynomial of nth degree:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Then, its coefficients can be expressed by means of the values of the polynomial itself and its derivatives, at the point x, by means of the formulae

$$a_i = \frac{P_n^{(i)}(0)}{i!} \quad (i = 0, 1, 2, \ldots, n),$$

$$P_n^{(0)}(0) = P_n(0),$$

and, therefore, the polynomial can be written down in the form

$$P_n(x) = P_n(0) + \frac{P_n'(0)}{1!}x + \frac{P_n''(0)}{2!}x^2 + \cdots + \frac{P_n^{(n)}(0)}{n!}x^n$$
$$= \sum_{i=0}^{n} \frac{P_n^{(i)}(0)}{i!}x^i. \tag{1.39}$$

If the polynomial $P_n(x)$ is expanded in terms of powers of the difference $(x - x_0)$:

$$P_n(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \cdots + A_n(x - x_0)^n,$$

then

$$A_i = \frac{P_n^{(i)}(x_0)}{i!} \quad (i = 0, 1, 2, \ldots, n), \quad P_n^{(0)}(x_0) = P_n(x_0).$$

Thus,

$$P_n(x) = P_n(x_0) + \frac{P_n'(x_0)}{1}(x - x_0) + \frac{P_n''(x_0)}{2!}(x - x_0)^2 + \cdots$$
$$+ \frac{P_n^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{i=0}^{n} \frac{P_n^{(i)}(x_0)}{i!}(x - x_0)^i. \tag{1.40}$$

The formula (1.40) as well as its particular case (1.39) is called *Taylor's formula* for a polynomial. Formula (1.39) is also often called *Maclaurin's formula.*

Suppose, now, that $f(x)$ is an arbitrary function, defined in the interval $[a, b]$ and possessing n continuous derivatives, $f'(x)$, $f''(x)$, ..., $f^{(n)}(x)$. Then, in the neighbourhood of any point $x = x_0 \in (a, b)$ for $x_0 + h \in (a, b)$ the following expansion holds:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \cdots$$
$$+ \frac{f^{(n)}(x_0)}{n!}h^n + o(h^n). \tag{1.41}$$

If the function $f(x)$ has a derivative of the order $(n + 1)$ (not necessarily a continuous one) then the expansion can be written down, instead of (1.41), in the form

$$(fx_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \cdots$$
$$+ \frac{f^{(n)}(x_0)}{n!}h^n + O(h^{n+1}). \tag{1.42}$$

If in a neighbourhood of the point $x = x_0 \in (a, b)$ the function $f(x)$ is represented in the form

$$f(x_0 + h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n + o(h^n),$$

then this expansion is unique, there exist at the point $x = x_0$, n first differential derivatives (see § 2) and the coefficients a_i ($i = 0, 1, 2, \ldots, n$) satisfy the relationships

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{\tilde{f}''(x_0)}{2!}, \ldots, \quad a_n = \frac{\tilde{f}^{(n)}(x_0)}{n!}.$$

Formulae (1.41) and (1.42) are called Taylor's formulae. In particular, when $x_0 = 0$ they are also called Maclaurin's formulae. They may be rewritten differently, making use of differentials

$$\Delta f(x_0) \equiv f(x_0 + h) - f(x_0) = df(x_0, h) + \frac{1}{2!} d^2 f(x_0, h) + \cdots + \frac{1}{n!} d^n f(x_0, h) + o(h^n), \tag{1.43}$$

$$\Delta f(x_0) = df(x_0, h) + \frac{1}{2!} d^2 f(x_0, h) + \cdots + \frac{1}{n!} \times d^n f(x_0, h) + O(h^{n+1}). \tag{1.44}$$

In the case when the point x_0 is one of the ends of an interval $[a, b]$, the derivative functions of $f(x)$ are understood to be one-sided derivatives, whose existence is assumed. Here h should be such that $x_0 + h \in [a, b]$.

8. Suppose the function $f(x)$ is defined in some set $X \subset E_1$. This function is called *infinitely differentiable* at the point $x = x_0 \in X$, if it has finite derivatives of all orders at the point $x = x_0$. For a function, which is infinitely differentiable at every point of X, we can write, according to Taylor's formula (1.42):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x), \tag{1.45}$$

where $r_n(x) = O[(x - x_0)^{n+1}]$ is called a *remainder* or a *complementary term*. The number of terms n in the formula (1.45) can be

as large as desired. As a result of this, we arrive formally in the right-hand side of the equation (1.45) at the *power series*

$$f(x_0) + \frac{f'(x_0)}{1}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots, \tag{1.46}$$

which is called *Taylor's series of the function* $f(x)$. The coefficients of this series,

$$a_0 = f(x_0), \quad a_n = \frac{f^{(n)}(x_0)}{n!} \quad (n = 1, 2, \ldots), \tag{1.47}$$

are called *Taylor's coefficients* of the function $f(x)$.

The power series (1.46) corresponds to the function $f(x)$; however, in the general case it cannot be maintained that this series converges, and in the case of convergence, that its sum equals the function $f(x)$. If we denote the partial sum of the series (1.46) by $S_n(f)$:

$$S_n(f) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

then $f(x) - S_n(f) = r_n(x)$. It follows, hence, that *in order that Taylor's series* (1.46) *converge to function* $f(x)$ *at point* $x \in X$ *it is necessary and sufficient that the remainder* $r_n(x)$ *in Taylor's formula converges to zero at the given point* x, *when* $n \to \infty$:

$$\lim_{n\to\infty} r_n(x) = 0.$$

In fulfilling this condition, we have the expansion of an infinitely differentiable function into a power series (Taylor's series):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots \tag{1.48}$$

Thus, in order to investigate the possibility of representing a function by means of a power series it is necessary to investigate the

behaviour of the remainder in Taylor's formula. The simplest form, *Peano's form:*

$$r_n(x) = o[(x - x_0)^n],$$

has been quoted above (see (1.41)). Below, we quote various forms of the remainder containing the $(n+1)$th derivative:

(a) *Lagrange's form*

$$r_n(x) = \frac{f^{(n+1)}[x_0 + \vartheta(x - x_0)]}{(n+1)!}(x - x_0)]^{n+1};$$

(b) *Cauchy's form*

$$r_n(x) = \frac{f^{(n+1)}[x_0 + \vartheta(x - x_0)]}{n!}(1 - \vartheta)^n (x - x_0)^{n+1};$$

(c) *Schlömilch's form* $(p > 0)$

$$r_n(x) = \frac{f^{(n+1)}[x_0 + \vartheta(x - x_0)]}{n!p}(1 - \vartheta)^{n+1-p}(x - x_0)^{n+1};$$

(d) *Integral form*

$$r_n(x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)\,(x - t)^n\,dt.$$

The factor ϑ, encountered in formulae (a), (b) and (c) satisfies the condition $0 < \vartheta < 1$. In addition ϑ depends on x, n and the form of the remainder. Lagrange's and Cauchy's forms for the remainder can be obtained as particular cases of the more general Schlömilch form, if we put $p = n + 1$ and $p = 1$ respectively in it.

A particular case of expansion of (1.48) which is called *Maclaurin's series*, is widely applicable; it is obtained from (1.48) when $x_0 = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \quad (1.49)$$

The coefficients of Maclaurin's series can be obtained from formulae (1.47) putting $x_0 = 0$:

$$a_0 = f(0), \quad a_n = \frac{f^{(n)}(0)}{n!} \quad (n = 1, 2, \ldots). \qquad (1.50)$$

The representation of a function by means of a power series is

unique. Namely, if a function $f(x)$ is represented by means of a power series converging to it in some interval

$$f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k,$$

$x_0 \in X$, then $f(x)$ is infinitely differentiable in X, this series is its Taylor's series and the coefficients of this series are connected with (fx) by means of formulae (1.47).

A function $f(x)$ which allows the expansion into power series converging to the function in the neighbourhood of point $x = x_0 \in X$ or, which is the same, a function whose Taylor's series converges to it in the neighbourhood of point $x = x_0 \in X$, is said to be *analytical at point* $x = x_0$. If $f(x)$ is analytical at the point $x = x_0 \in X$, it is *analytical in some interval* containing point x_0.

A function $f(x)$ which is analytical at the point $x = x_0 \in X$ is infinitely differentiable at that point. The converse conclusion does not hold. Even in the case when Taylor's series converges, its sum may differ from $f(x)$, i.e. the function does not expand into a power series converging to it.

For example,

$$f(x) = \begin{cases} e^{-1/x^2} & \text{when} \quad x \neq 0, \\ 0 & \text{when} \quad x = 0 \end{cases}$$

is infinitely differentiable for all $x \in (-\infty, +\infty)$. For $x_0 = 0$, all its derivatives become zero. Therefore Taylor's series for this function converges everywhere, but its sum equals zero everywhere (all coefficients of the series equal zero) and does not coincide with the function $f(x)$. Therefore, the function $f(x)$ cannot be expanded into a power series converging to it in a neighbourhood of the point $x = x_0$ and is not analytic.

§ 3. The Application of Derivatives in Investigating Functions. Extrema

1. Suppose the function $y = f(x)$ is continuous in the interval $[a, b]$ and has a derivative (finite or infinite) at all its points, except perhaps, the end-points. We have the following theorem connecting the character of the variation of the function with the sign of its derivative:

THEOREM 13. *In order that a function $f(x)$ be monotonically non-decreasing*† *(non-increasing) it is necessary and sufficient that its*

† Let us recall that a function is called *monotonically non-decreasing (non-increasing)* in the interval $[a, b]$, if, for any points x_1 and x_2 belonging to $[a, b]$ and such that $x_1 < x_2$, the inequality $f(x_1) \leqslant f(x_2)$ $(f(x_1) \geqslant f(x_2))$ holds. A function is called *monotonically increasing (decreasing)* in the strict sense, if in the same conditions $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$.

derivative is non-negative (non-positive):

$$f'(x) \geqslant 0 \quad (f'(x) \leqslant 0).$$

Geometrically, this means, that the tangent to the graph of the non-decreasing (non-increasing) function is not inclined to the axis of the abscissae at an obtuse (acute) angle at any point.

THEOREM 14. *In order that a function $f(x)$ be monotonically increasing (decreasing) in the strict sense, it is sufficient that its derivative be non-negative (non-positive):*

$$f'(x) \geqslant 0 \quad (f'(x) \leqslant 0)$$

and that it does not become zero identically in any interval within $[a, b]$.

For example, the function $y = x^3$ has the derivative $y' = 3x^2$, which is positive everywhere except the point $x = 0$, where it becomes zero. This function increases monotonically in any segment of the numerical axis (in short, over the whole numerical axis).

The function $y = x - \sin x$ has the derivative $y' = 1 - \cos x$, which turns into zero at points $x = \pi/2 + 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ and is positive at all other points. Therefore, this function is an increasing one over the whole numerical axis.

In application to inequalities, it follows from the theorems formulated, that if $f'(x) > 0$ in the interval (a, b) and $f(a) \geqslant 0$, then $f(x)$ is positive in the interval (a, b). For example, for the function $f(x) = x - \ln(1 + x)$ we have $f(0) = 0$ and $f'(x) = 1 - 1/(1 + x) > 0$, for $x > 0$. Therefore, $x > \ln(1 + x)$ for all $x > 0$.

In the case, when the function $f(x)$ does not have a definite derivative at all points of the interval (a, b), the formulated signs can allow the following generalization to be made:

THEOREM 15. *If the function $f(x)$ is continuous in $[a, b]$ and one of its generalized derivatives (see § 1, sec. 9) remains negative (non-positive) all the time, without, however, becoming zero in any interval belonging to $[a, b]$, then the function $f(x)$ strictly increases (decreases) in the interval $[a, b]$.*

2. The local behaviour of a function in the inner points of the set X is characterized by the following theorem.

THEOREM 16. *If $f'(x_0) > 0$ $(f'(x_0) < 0)$, there can be found a neighbourhood of the point x_0 such that for all the points of this neighbourhood to the right of x_0*

$$f(x) > f(x_0) \quad (f(x) < f(x_0)),$$

and for all points to the left of x_0

$$f(x) < f(x_0) \quad (f(x) > f(x_0)).$$

In this case, it is said, that the function $f(x)$ *increases (decreases) at the point* x_0.

It should be kept in mind that here the function $f(x)$ is *not necessarily monotonic* in any of the intervals surrounding the point x_0.

Thus, for the function

$$y = x^2 \sin\frac{1}{x} + \alpha x \qquad (0 < \alpha < 1)$$

the derivative at point $x = 0$ equals (see example on p. 16)

$$y'_{x=0} = \alpha > 0.$$

At the same time at other points

$$y' = 2x \sin\frac{1}{x} - \cos\frac{1}{x} + \alpha,$$

and this expression has, for $x \to 0$, lower and upper limits equal to $\alpha - 1$ and $\alpha + 1$ respectively. Since $\alpha - 1 < 0$ and $\alpha + 1 > 0$, therefore within a short distance from the point x_0 there can be found points in which $y' < 0$ and points in which $y' > 0$. It follows that there does not exist an interval containing the point $x = 0$ in which the function is monotonic.

3. Definition. It is said that the continuous function $f(x)$ has a *maximum (minimum)* in an inner point x_0 of the set X, if there exists a neighbourhood of the point x for all of whose points the following inequality holds:

$$f(x) < f(x_0) \quad (f(x) > f(x_0)).$$

The point x_0 in this case is called *the maximum (minimum) point of the function.* The values of the function at these points are called *extremal values of the function.* Thus, in Fig. 4, the points x_1 and x_3 are maximum points and points x_2 and x_4 are minimum points.

The inequalities shown define *strict extremal points* (they are usually known simply as extremal points).

If we exchange the signs $<$ and $>$ for $\leqslant$ and $\geqslant$, then the point x_0 is a non-strict extremal point.

The definition of extremal points is of a localized nature. A function may have any number of maxima and minima, while some of its minimal values may turn out to be greater than some of the maximal ones (in Fig. 4, $f(x_4) > f(x_1)$).

A NECESSARY CONDITION FOR AN EXTREMUM. *A function $f(x)$ can have an extremum only at those points at which its derivative $f'(x)$ either becomes zero or does not exist.*

The geometric illustration of various possible cases is represented in Fig. 5. At the points x_1 and x_2 the extremum is a smooth one, at these points the tangent to the graph of the function is parallel

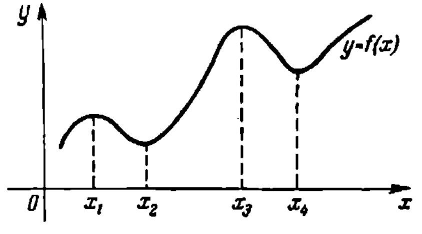

FIG. 4

to the axis of the abscissae. The points x_3, and x_4 are cusps with a vertical tangent; the points x_5 and x_6 are nodes of the graph.

Examples such as the functions $y = x^3$ and $y = \sqrt[3]{x}$, the first of which has a derivative equal zero, when $x = 0$, and the second

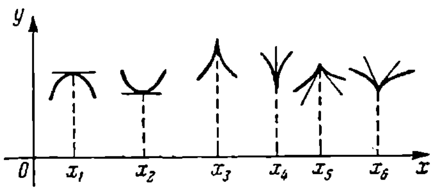

FIG. 5

has no derivative, when $x = 0$ ($y' = +\infty$ when $x = 0$), show that the necessary sign is not also sufficient.

FIRST SUFFICIENT CONDITION. *Suppose the continuous function $f(x)$ has a first derivative $f'(t)$ in some neighbourhood of point x_0, excluding, perhaps, the point x_0 itself, and the derivative does not change sign either for $x < x_0$ or for $x > x_0$, where x is a point in the neighbourhood mentioned.*

Then, if the derivative is positive (negative) for $x < x_0$ and negative (positive) for $x > x_0$, the point x_0 is a maximal (minimal) point of the function $f(x)$.

At the point x_0 itself, the derivative equals zero, or does not exist, in accordance with the necessary sign of an extremum.

This rule can be briefly formulated as follows.

If the derivative $f'(x)$, on passing the point x_0, changes its sign from plus to minus, the point x_0 is a maximum, and if its sign changes from minus to plus, the point x_0 is a minimum.

If the derivative $f'(x)$ keeps the same sign both on the left and on the right of point x_0, the point x_0 cannot be an extremal point.

From the geometric point of view the first sufficient rule means that if the interval in which the function increases is succeeded by an interval in which it decreases (or conversely) then the point separating these intervals is a maximum (a minimum).

The converse, however, does not always hold, i.e. an extremal point does not necessarily separate intervals of monotony of opposite sense of a function.

Thus, for example, the function

$$f(x) = \begin{cases} x^2\left(\sin^2 \dfrac{1}{x} + \alpha\right) & \text{when} \quad x \neq 0, \ \alpha > 0, \\ 0 & \text{when} \quad x = 0 \end{cases}$$

is positive for $x \neq 0$, therefore, the point $x = 0$ is a minimum. A derivative at this point exists and equals zero:

$$\lim_{h \to 0} \frac{h^2\left(\sin^2 \dfrac{1}{h} + \alpha\right)}{h} = 0.$$

However, in neighbouring points, the derivative

$$f'(x) = 2x\left(\sin^2 \frac{1}{x} + \alpha\right) - \sin \frac{2}{x}$$

has, for $x \to 0$, as upper and lower limits, $+1$ and -1 respectively, i.e. anywhere close to the point x_0 there exist points where $f'(x) > 0$ and points where $f'(x) < 0$; this means that no interval ending at the point $x = 0$ is an interval of monotony. (The construction of this example is analogous to the construction of the example in sec. 2.)

SECOND SUFFICIENT CONDITION. *Suppose, at the point x_0, we have $f'(x_0) = 0$. Let us work out the successive derivatives $f^{(k)}(x_0)$, provided they exist. If the first derivative $f^{(k)}(x_0)$, which does not equal zero, is of an odd order, then the point x_0 is not an extremum; if, on the other hand, this derivative is of an even order, then for $x = x_0$ there is a maximum if it is negative, and a minimum if it is positive.*

In practice, this rule is used when $f''(x_0) \neq 0$.

Note that the first rule presupposes knowledge of the behaviour of the first derivative in the neighbourhood of the point x_0; the

second rule requires only the knowledge of particular values of certain successive derivatives.

In investigating the function $f(x)$ for an extremum, the following note is often useful. If a is the λ-fold root of the derivative $f'(x)$, it can be represented in the form

$$f'(x) = (x - a)^{\lambda}\varphi(x), \tag{1.51}$$

where $\varphi(x)$ does not become zero at the point a. The point a is an extremum if λ is an odd number, and it is not an extremum if λ is an even number. In the first case, the point a is a maximum when $\varphi(a) < 0$, and a minimum when $\varphi(a) > 0$.

The point a is also an extremum, if in the representation (1.51) has the form p/q, where p and q are odd (the fraction p/q is in its lowest terms).

EXAMPLE 12. If $y = x^n e^{-x}$ (n is a natural number) the derivative y' equals $y' = x^{n-1}e^{-x}(n - x)$.

In accordance with the preceding note, the point $x = n$ for any n is an extremum and a maximum at that. The value of the function at that point is $y|_{x=n} = \left(\frac{n}{e}\right)^n$.

The point $x = 0$ is extremal *only* when n is even; in this case it is a minimum. When n is odd, the point $x = 0$ is not an extremum.†

EXAMPLE 13. The derivatives of the function $y = e^x \sin x$ equal:

$$y' = e^x(\sin x + \cos x), \quad y'' = 2e^x \cos x.$$

The derivative y' becomes zero when $\sin x + \cos x = 0$, i.e. when $\tan x = -1$. Here $x = \pi n - (\pi/4)$ ($n = 0, \pm 1, \pm 2, \ldots$).

Let us substitute these values in the second derivative.

Since $\cos\,[2k\pi - (\pi/4)] = \cos(\pi/4) > 0$ and $\cos\,[(2k+1)\pi - (\pi/4)] = -\cos(\pi/4) < 0$, therefore, according to the second sufficient sign, the points $x = 2k\pi - (\pi/4)$ are minima and the points $x = (2k+1)\pi - (\pi/4)$ are maxima ($k = 0, \pm 1, \pm 2, \ldots$).

4. Suppose we are given a continuous function $f(x)$ in the interval $[a, b]$. According to the properties of continuous functions it reaches its *absolute maximum and minimum* or, in other words, its *greatest and least value* in this interval. The finding of these is usually based on the investigation of the function for an extremum, so the

† The point O(0, 0) turns out to be a point of *inflection*.

function $f(x)$ can reach its greatest (least) value in the interval $[a, b]$ either at its maxima (minima) or at the ends of the interval.

Figure 6 shows various possible cases.

The finding of the greatest and the least values of the function is carried out particularly simply in two cases:

(a) If the function $f(x)$ increases (decreases) monotonically in the interval $[a, b]$, it has its least value at the left (right) end of the interval, and its greatest value at the right (left) end of that interval.

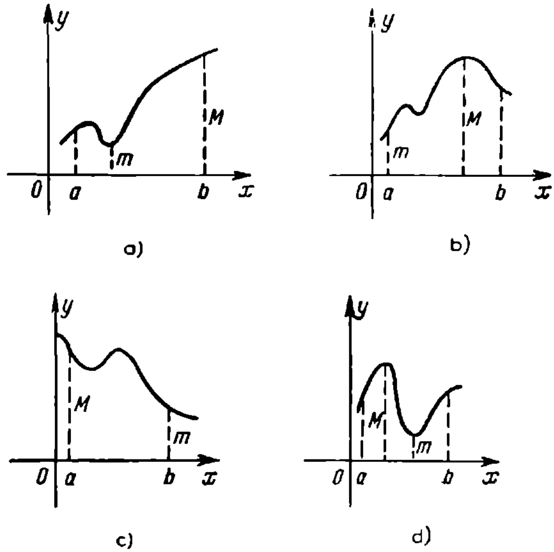

FIG. 6

(b) If a function has only one extremal point in the interval (a, b), then the function reaches its greatest value at that point if the point is a maximum, and its least value if the point is a minimum.

EXAMPLE 14. Finding the greatest (M) and the least (m) value of the function $y = x/(1+x^2)$ in the interval $[\frac{1}{2}, 3]$.

The derivative $y' = (1-x^2)/(1+x^2)^2$ becomes zero in this interval at the point $x = 1$. Comparing the value of the function at this point and at the ends of the intervals

$$y|_{x=1/2} = \tfrac{2}{5}, \quad y|_{x=1} = \tfrac{1}{2}, \quad y|_{x=3} = \tfrac{3}{10},$$

we find that $M = \frac{1}{2}$ and $m = \frac{3}{10}$.

EXAMPLE 15. We are given the function $y = \sin 2x - x$ in the interval $[-\pi/2, \pi/2]$. The derivative $y' = 2\cos 2x - 1$ becomes zero in the given interval, when

$x = \pm\pi/6$. On comparing the values of the function at the points $x = \pm\pi/6$ and at the points $x = \pm\pi/2$:

$$y|_{x=\pm\pi/6} = \pm\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) \approx \pm 0{\cdot}34, \quad y|_{x=\pm\pi/2} = \mp\frac{\pi}{2} \approx \mp 1{\cdot}57,$$

we find that $M = 1{\cdot}57$ and $m = -1{\cdot}57$. Both values are attained by the function at the ends of the interval.

5. The establishment of criteria of convexity† of a function is also based on the application of derivatives.

THEOREM 17. *If the function $f(x)$ has a first derivative $f'(x)$ at all points of the interval $[a, b]$, then in order that the function $f(x)$ be convex (convex upwards) it is necessary and sufficient that $f'(x)$ be a non-decreasing (non-increasing) function.*

As a corollary, we obtain the following.

THEOREM 18. *If the function $f(x)$ has a second derivative $f''(x)$ at all points of the interval $[a, b]$, then in order that the function $f(x)$ be convex (convex upwards) it is necessary and sufficient that $f''(x) \geqslant 0$ ($f''(x) \leqslant 0$) for any x in the interval $[a, b]$.*

If the function $f(x)$, convex in the interval $[a, b]$, does not have an ordinary derivative at all points, the following *properties of convex functions* take place.

(1) At any point of the interval (a, b) there exist a right-hand derivative and a left-hand derivative, $f'_+(x)$ and $f'_-(x)$, and

$$f'_-(x) \leqslant f'_+(x).$$

(2) The derivative $f'_+(x)$ is a non-decreasing function, continuous on the right.

(3) The derivative $f'_-(x)$ is a non-decreasing function, continuous on the left.

The properties of functions convex upwards can be formulated analogously.

† The continuous function $f(x)$ is called *convex downwards* or simply *convex* in the interval $[a, b]$, if for any two points x_1 and x_2 of this interval

$$f\left(\frac{x_1 + x_2}{2}\right) \leqslant \frac{f(x_1) + f(x_2)}{2}.$$

If the inequality holds in the opposite sense, i.e.

$$f\left(\frac{x_1 + x_2}{2}\right) \geqslant \frac{f(x_1) + f(x_2)}{2},$$

the function is called *convex upwards*. Often we may come across a different terminology, namely, it is said, that in the first case the graph of the function is convex downwards, i.e. *concave*, and in the second case the graph is convex upwards, i.e. *convex*.

§ 4. Differential Operators

1. In various branches of analysis, for example, in the calculus of variations, one has to consider classes (sets) of functions satisfying different sets of conditions.

We denote the class (set) of functions $f(x)$ which are defined and continuous in the set X by $C = C[X]$. The symbol $C_n = C_n[X]$, denotes a class of functions $f(x)$ defined in the set X and differentiable continuously n times in X. The latter means (see § 2) that $f(x)$ is continuous in X and has the derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$, which are also continuous in X.

For example, the function $f(x) = e^x$ belongs to any class C_n; the function $|x|$ belongs to the class $C[-1, +1]$, but does not belong to the class $C_1[-1, +1]$; the function $|x|^{2n+1}$ belongs to the class $C_{2n}[-1, +1]$, but does not belong to the class $C_{2n+1}[-1, +1]$.

All the classes mentioned are *linear systems*, i.e. (1) as well as the functions f_1 and f_2 belonging to a given class there belongs to this class also the function $f_1 + f_2$ and (2) for any real number λ, the function λf belongs to the same class as the function f.

2. Let X and Y be two arbitrary sets (for example, classes of functions) and suppose a law (rule) is given, according to which to each element $x \in X$ there corresponds a unique, completely defined element $y \in Y$. Then it is said, that we are given an *operator* (or *abstract function*) $y = Ax$ (or $y = A(x)$) defined in the set X. Here the set X is called the *domain of definition of the operator* A, the element $y = A(x)$ is called the image of the element x, the set $Y_A \subset Y$ of all images of the elements of X is called the range of the operator. For each element $y \in Y$, the element $x \in X$, for which $A(x) = y$ is called the *inverse image* or *original* of this element.

Suppose, the sets X and Y are linear systems; then the operator $y = Ax$, mapping X into Y is said to be *linear*, if it is

(a) *additive*, i.e. for any x_1 and x_2 belonging to X

$$A(x_1 + x_2) = Ax_1 + Ax_2; \tag{1.52}$$

(b) *homogeneous*, i.e. for any real number λ

$$A(\lambda x) = \lambda A(x). \tag{1.53}$$

3. The linear operator for the case, when X and Y are finite-dimensional systems, was considered in Chapter II of volume 69 of

this series. In this chapter, we discuss linear operators acting on functions $f(x)$ of a certain class.

An example of such a linear operator is provided by the operator $A = [x]$ of multiplication by the argument

$$Af(x) \equiv [x]f(x) = xf(x).$$

Another example of a linear operator is the operator of differentiation $D = d/dx$ (see formulae (1.2) and (1.3)).

The name *operator of differentiation* $= d/dx$ is given to the operator, which causes each function $\varphi(x) = f(x) = Df'(x)$ *(the derivative)* of the class $C[X]$, to correspond to every function $f(x)$ of the class $C_1[X]$; i.e. it converts any function $f(x) \in C_1$ into some function (derivative) $f'(x) \in C$.

Thus, $D = d/dx$ is an *operator mapping the class* C_1 *into the class* C.

Similarly, the operator $D^n = d^n/dx^n$ transforms any function $f(x)$ of the class C_n into a function $\varphi(x) = D^n f(x) = f^{(n)}(x)$ of the class C, i.e. *the operator* $D^n = d^n/dx^n$ *maps the class* C_n *into the class* C.

Suppose $P_n(t)$ is a polynomial of degree n with real coefficients

$$P_n(t) = \sum_{k=0}^{n} a_k t^k. \tag{1.54}$$

$P_n(D)$ denotes the operator

$$P_n(D) = \sum_{k=0}^{n} a_k D^k; \tag{1.55}$$

it is called a *differential polynomial* or a polynomial of *the operator of differentiation.*

The operator

$$P_n(D) = \sum_{k=0}^{n} a_k D^k$$

maps the class C_n into the class C, i.e. for any function $f(x)$ belonging to the class C_n, the function

$$P_n(D)f(x) = \sum_{k=0}^{n} a_k f^{(k)}(x)$$

belongs to the class C.

Some properties of differential polynomials:

(a) $$P_n(D)\, e^{\lambda x} = P_n(\lambda)\, e^{\lambda x}. \tag{1.56}$$

(b) *A generalization of Leibniz's formula.* If $P_n(t) = \sum_{k=0}^{n} a_k t^k$, then

$$P_n(D)\,(u_1u_2) = \sum_{k=0}^{n} a_k D^k(u_1u_2) = [P_n(D)u_1]u_2 + [P_n'(D)u_1]Du_2$$

$$+ \frac{1}{2!}[P_n''(D)u_1]D^2u_2 + \frac{1}{3!}[P_n'''(D)u_1]D^3u_2 + \cdots + \frac{1}{k!} \times$$

$$\times [P_n^{(k)}(D)u_1]D^ku_2 + \cdots + \frac{1}{n!}[P_n^{(n)}(D)u_1]D^nu_2, \tag{1.57}$$

or, which is shorter,

$$P_n(D)\,(u_1u_2) = \sum_{k=0}^{n} \frac{1}{k!}[P_n^{(k)}(D)u_1]D^ku_2, \tag{1.58}$$

where $P_n^{(k)}(t)$ denotes the kth derivative of the polynomial $P_n(t)$.

From (a) and (b) it follows, that

(c) $$P_n(D)\,(x^m e^{\lambda x}) = \{x^m P_n(\lambda) + mx^{m-1}P_n'(\lambda) + \cdots$$
$$+ C_m^k x^{m-k} P_n^{(k)}(\lambda) + \cdots + P_n^{(m)}(\lambda)\}\, e^{\lambda x}. \tag{1.59}$$

EXAMPLE 16. When $P_3(t) = t^3 - 2t + 1$, we have

$$P_3'(t) = 3t^3 - 2, \quad P_3''(t) = 6t, \quad P_3'''(t) = 6, \quad P_3^{\mathrm{IV}}(t) = 0.$$

Therefore

$$P_3(D)e^{\lambda x} \equiv \left(\frac{d^3}{dx^3} - 2\frac{d}{dx} + 1\right) e^{\lambda x} = (\lambda^3 - 2\lambda + 1)e^{\lambda x}.$$

In the same way

$$P_3(D)\,(x^4e^{\lambda x}) \equiv \left(\frac{d^3}{dx^3} - 2\frac{d}{dx} + 1\right)(x^4e^{\lambda x})$$

$$= \{x^4P_3(\lambda) + 4x^3P_3'(\lambda) + 6x^2P_3''(\lambda) + 4xP_3'''(\lambda)\}\, e^{\lambda x}$$

$$= \{x^4(\lambda^3 - 2\lambda + 1) + 4x^3(3\lambda^2 - 2) + 36x^2\lambda + 24x\}\, e^{\lambda x}.$$

(d) If the number λ_0 is a k-fold root of the polynomial $P_n(t)$, i.e.

$$P_n(\lambda_0) = P_n'(\lambda_0) = \cdots = P_n^{(k-1)}(\lambda_0) = 0, \quad P_n^{(k)}(\lambda_0) \neq 0,$$

the following equations hold

$$P_n(D)\, e^{\lambda_0 x} = P_n(D)\,(xe^{\lambda_0 x}) = \cdots = P_n(D)\,(x^{n-1}e^{\lambda_0 x}) = 0. \tag{1.60}$$

EXAMPLE 17. $\lambda = 1$ is the triple root of the polynomial

$$P_4(t) = t^4 - 3t^3 + 3t^2 - t = t(t-1)^3,$$

as a consequence of which

$$P_3(D)e^x = P_3(D)(xe^x) = P_3(D)(x^2e^x) = 0.$$

(e) If

$$P_n(t)P_m(t) \equiv P_{n+m}(t) = P_m(t)P_n(t),$$

then also

$$P_n(D)P_m(D) = P_{n+m}(D) = P_m(D)P_n(D). \tag{1.61}$$

EXAMPLE 18. For $P_1(t) = t + 1$, $P_2(t) = t^2 - t + 1$, we find

$$P_{1+2}(t) = P_1(t)P_2(t) = (t+1)(t^2 - t + 1)$$
$$= t^3 + 1 = (t^2 - t + 1)(t+1),$$

therefore

$$P_1(D)P_2(D) = (D+1)(D^2 - D + 1) = P_2(D)P_1(D)$$
$$= (D^2 - D + 1)(D+1) = D^3 + 1 = P_3(D).$$

4. The set of operators acting from the set E to the set E_1 forms *a linear system*, if for any two such operators A and B it is possible to define their sum $A + B$ as an operator from E to E_1 and for any operator A it is possible to define *the product of the operator A and a real constant* λ—operator λA—acting from E to E_1. Here, these operations on the operators should be defined in such a way, that for any element x belonging to E, the following conditions are fulfilled:

$$(A + B)x = Ax + Bx \in E_1, \tag{1.62}$$

$$(\lambda A)x = \lambda Ax \in E_1. \tag{1.63}$$

Suppose, further, that the operator A maps E_1 into E and the operator B maps E_2 into E_1. Then *the product AB of operators A and B* is understood to be the operator, acting from E_2 to E, and we have, for any element $x \in E_2$,

$$AB(x) = A(Bx). \tag{1.64}$$

Thus, *the product of operators AB means the consecutive application first of operator B and then of operator A.*

If $E \subset E_1 \subset E_2$, the operator A maps E into E_1 and E_1 into E_2, then the operator $A^2 = A \times A$ maps E_2 into E. In particular, $D^2 = D \times D$ is the operator mapping the class C_2 into the class C.

Similarly, we define *the product of n operators* $A_nA_{n-1}, \ldots, A_2A_1$ as the consecutive application of operators $A_1, A_2, \ldots, A_{n-1}, A_n$, and *the n-th power* A^n *of the operator* A:

$$A^n = \underbrace{A \,.\, A \,.\, \cdots \,.\, A}_{n \text{ times}}.$$

(For example, the operators $P_n(D) = \sum_{k=0}^{n} a_k D^k$, considered above, are obtained by raising D to the power k ($k \leqslant n$), multiplying by constants a_k ($k = 0, 1, \ldots, n$) and adding up.)

If the operators BA and AB are defined and $BA = AB$, i.e. for any x, for which this expression has a meaning, the following equation holds

$$A(Bx) = B(Ax), \tag{1.65}$$

the operators A and B are said to *commute with each other* or to be permutable. For example, any operators $P_n(D)$ and $P_m(D)$ commute as can be seen from (1.61).

The operator $[x]$ of multiplication by the argument (see section 3) does not commute with the operator D. Indeed, for the function $f(x)$ of the class C_1 we have

$$[x]\,(Df(x)) = [x]f'(x) = xf'(x);$$

on the other hand

$$D([x]f(x)) = D(xf(x)) = (xf(x))' = f(x) + xf'(x).$$

Thus, $[x]Df(x) \neq D[x]f(x)$. From the expressions for the left and right sides we have

$$(D[x] - [x]D)f(x) = f(x). \tag{1.66}$$

Let us denote a unit operator, that does not alter the function $f(x)$ on which it acts, by I, i.e.

$$If(x) = f(x). \tag{1.67}$$

Then, it follows from (1.66), that in the class C_1

$$D[x] - [x]D = I. \tag{1.68}$$

Similarly

$$P_n(D)\,[x] - [x]\,P_n(D) = P_n'(D). \tag{1.69}$$

The Linear Differential Operator

5. Every operator of the form

$$A = \sum_{k=0}^{n} [\psi_k(x)]D^k, \tag{1.70}$$

where $[\varphi_k(x)]$ is the operator of multiplication by a continuous function $\psi_k(x)$, ($\psi_n(x) \not\equiv 0$ is a continuous function in the set X), is called a *linear differential operator*. The number n is called *the order of the operator*. The linear differential operator (1.70) maps any function from the class C_n into some function of the class C:

$$Af(x) = \sum_{k=0}^{n} [\psi_k(x)]D^k f(x) = \sum_{k=0}^{n} \psi_k(x) f^{(k)}(x). \tag{1.71}$$

Every linear differential operator

$$B = \sum_{k=n}^{n+m} [\psi_k(x)]D^k$$

converts any polynomial

$$P_{n-1}(x) = \sum_{s=0}^{n-1} a_s x^s$$

of degree $n - 1$ into zero. Indeed,

$$\begin{aligned} BP_{n-1}(x) &= \sum_{k=n}^{n+m} \psi_k(x) D^k P_{n-1}(x) = \sum_{l=0}^{m} \psi_{n+l}(x) D^{n+l} P_{n-1}(x) \\ &= \sum_{l=0}^{m} \psi_{n+l} \frac{d^{n+l}}{dx^{n+l}} P_{n-1}(x) = 0. \end{aligned}$$

When the independent variable t is exchanged for the variable x by means of the substitution $t = \alpha(x)$ (and $\alpha \neq 0$), the differential operator $A(D_t)$ with respect to the variable t becomes the differential operator

$$\tilde{A}(D) \quad \left(\text{where} \quad D = D_x = \frac{d}{dx}\right)$$

with respect to the variable x of the same order as the operator D_t, provided that $\alpha(x) \in C_n$. In other words *the order of a differential operator does not alter when the independent variable is changed.*

CHAPTER II

THE DIFFERENTIATION OF FUNCTIONS OF n VARIABLES

§ 1. Derivatives and Differentials of the First Order

1. Let $X(x_1, x_2, \ldots, x_n)$ or $X = (x_1, x_2, \ldots, x_n)$ denote an element of n-dimensional space E_n, which can be treated as a point or as a vector with coordinates $x_1, x_2, \ldots, x_n$. e_i $(i = 1, 2, \ldots, n)$ denote unit vectors in E_n in the direction of the x_i-axes:

$$X(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i e_i.$$

The function $f(X) = f(x_1, x_2, \ldots, x_n)$ is a function of n-variables $x_1, x_2, \ldots, x_n$ or, which is the same, the function of the point (vector) $X(x_1, x_2, \ldots, x_n)$ in E_n.

The *norm* (Euclidean) of the vector $X(x_1, x_2, \ldots, x_n)$, $\|X\|$, is determined by the equation

$$\|X\| = \sqrt{\sum_{i=1}^{n} x_i^2}, \tag{2.1}$$

and the *distance* between the points $X(x_1, x_2, \ldots, x_n)$ and $Y(y_1, y_2, \ldots, y_n)$ is determined by the equation

$$\varrho(X, Y) = \|X - Y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{2.2}$$

A sphere $S(X^0, r)$ of radius $r\,(> 0)$ and centre at the point X^0 is a set of points X, for which $\varrho(X^0, X) = \|X - X^0\| < r$. The set of points $\{X_t = X^0 + th\}$, $-\infty < t < +\infty$, is a straight line joining the points $X^0(x^0, x^0, \ldots, x^0)$ and $X^0 + h$, where $h = (h_1, h_2, \ldots, h_n)$, in E_n. The same set, when $0 \leqslant t \leqslant 1$, is called *a segment* joining points X^0 (the beginning) and $X^0 + h$ (the end). A finite set of seg-

ments in which the beginning of each segment coincides with the end of the preceding one is called an *open polygon.*

The region G in E_n is the set of points which contains both the point X^0 and some sphere $S(X^0, r)$. Here the region is supposed to be connected, i.e. any two of its points can be joined by means of an open polygon, all of whose points are situated within the region.

The name *neighbourhood* of the point X^0 in E_n is given to any region in E_n containing X^0. Each neighbourhood of the point X^0 contains a certain sphere $S(X^0, r)$.

2. Let $X(x_1, x_2, \ldots, x_n)$ and $X'(x_1', x_2', \ldots, x_n') = X + h_i e_i$ be points of E_n; evidently

$$x_j' = x_j \qquad \text{when} \quad j \neq i,$$
$$x_i' = x_i + h_i,$$

i.e. the ith coordinate has acquired the increment h_i.

If $f(X) = f(x_1, x_2, \ldots, x_n)$ is a function defined at points X, and $X' = X + h_i e_i$, the difference

$$\Delta_{h_i}^{x_i} f(X) = f(X + h_i e_i) - f(X) \tag{2.3}$$

is called *partial increment*, and the operator $\Delta_{h_i}^{x_i}$ is called the *operator of partial increment* corresponding to the increment h_i of the variable x_i.

For example, for the case of the plane E_2 (two variables)

$$\Delta_{h_1}^{x} f(X) = \Delta_{h_1}^{x} f(x, y) = f(x + h_1, y) - f(x, y),$$
$$\Delta_{h_2}^{y} f(X) = \Delta_{h_2}^{y} f(x, y) = f(x, y + h_2) - f(x, y).$$

The name *partial derivative* $(\partial/\partial x_i) f(X^0)$ of the function $f(X) = f({}_1x, x_2, \ldots, x_n)$ with respect to the variable x_i $(i = 1, 2, \ldots, n)$ at the point $X^0(x_1^0, x_2^0, \ldots, x_n^0)$ is given to the limit

$$\frac{\partial}{\partial x_i} f(X^0) = \lim_{h_i \to 0} \frac{F(X^0 + h_i e_i) - F(X^0)}{h_i} = \lim_{h_i \to 0} \frac{\Delta_{h_i}^{x_i} F(X^0)}{h_i}, \tag{2.4}$$

if it exists. For example, for the function of two variables $f(X) = f(x, y)$ the partial derivatives at the point $X^0(x_0, y_0)$ are

$$\left.\begin{aligned} \frac{\partial}{\partial x} f(x_0, y_0) &= \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1}, \\ \frac{\partial}{\partial y} f(x_0, y_0) &= \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{h_2}. \end{aligned}\right\} \tag{2.5}$$

If we fix all the coordinates x_j of the points $X(x_1, x_2, \ldots, x_n)$ when $j \neq i$, then the function $f(X)$ becomes a function of one variable x_i and $(\partial/\partial x_i)f(x_1, x_2, \ldots, x_n)$ is the derivative of this function with respect to the variable x_i.

The following definitions are introduced in order to facilitate the formulation of theorems.

It is said that the function $f(X)$, defined in the region G, belongs to the class C_G, if $f(X)$ is continuous in G. The function $f(X)$ belongs to the class $C_1 = C_{1,G}$, if at every point of G all partial derivatives $(\partial/\partial x_i)f(X)$ $(i = 1, 2, \ldots, n)$ are defined, and $(\partial/\partial x_i)f(X)$, as functions of X, are continuous in G.

In the class of functions $C_{1,G}$ there are defined *the operators of partial differentiation* $D_i = (\partial/\partial x_i)$ $(i = 1, 2, \ldots, n)$ which map $f(X) \subset C_{1,G}$ into the function

$$D_i f(X) = \frac{\partial}{\partial x_i} f(x)$$

belonging to C_G. These operators are linear.

3. Both definitions of the differential of a function of one variable can be generalized to include the case of n variables. Suppose the function $f(X) = f(x_1, x_2, \ldots, x_n)$ is defined at the point $X^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in E_n$ and in its neighbourhood, i.e. for all points of the form $X^0 + h$, where $h = (h_1, h_2, \ldots, h_n)$ and $\|h\| < r$ (r is a certain fixed positive number).

(a) *The differential as the principal linear part of the increment of the function.* The increment of function $f(X)$

$$f(X + h) - f(X) = f(x_1 + h_1, x_2 + h_2, \ldots, x_n + h^n) - f(x_1, x_2, \ldots, x_n) \tag{2.6}$$

corresponds to the vector-increment $h(h_1, h_2, \ldots, h_n)$ of the argument-vector X. For any vector h, where $0 < \|h\| < r$, let the increment (2.6) of the function f be capable of being represented in the form

$$f(X + h) - f(X) = L(h) + \varepsilon(h), \tag{2.7}$$

where

$$L(h) = \sum_{i=1}^{n} l_i h_i \tag{2.8}$$

is a linear function of the vector-increment $h(h_1, h_2, \ldots, h_n)$ and $\varepsilon(h)$ is a quantity of a greater order of smallness as compared with $\|h\|$:

$$\varepsilon(h) = o(\|h\|).$$

In this case $L(h)$ is called *the differential of the function* $f(X)$ at the point X and is denoted

$$L(h) = df(X) = df(X, h). \tag{2.9}$$

The properties of the differential df. 1°. If at the point $X^0(x_1^0, x_2^0, \ldots, x_n^0)$ the differential $df(X^0, h)$ exists, then there exist at this point all partial derivatives $(\partial/\partial x_i)f(X^0)$ $(i = 1, 2, \ldots n)$ and

$$df(X^0, h) = \sum_{i=1}^{n} \frac{\partial f(X^0)}{\partial x_i} h_i, \tag{2.10}$$

i.e. the coefficients l_i in (2.8) equal the partial derivatives $(\partial/\partial x_i)f$ at the point X^0.

It is said that the function $f(X)$ is *differentiable* at the point X^0, if a differential $df(X^0, h)$ exists at that point.

Sometimes $df(X^0, h)$ is called the *complete differential* of the function f at the point X^0, and the terms $[\partial f(X^0)/\partial x_i]\, h_i$ $(i = 1, 2, \ldots, n)$ its *partial differentials.* Since we have, for the function $f(X) = x_i$, $\partial f/\partial x_i = 1$, and the partial differential corresponding to the increment h_i of this coordinate equals $dx_i = h_i$, therefore, denoting the vector h with coordinates $dx_i = h_i$ by dX, we write down (2.10) in the form

$$df(X^0, dX) = \sum_{i=1}^{n} \frac{\partial f(X^0)}{\partial x_i} dx_i.$$

EXAMPLE 1. Let $X(x, y)$ be a point in a plane and $f(X) = f(x, y) = xy$. Then, if $X^0 = (x_0, y_0)$ and $h = (h_1, h_2)$,

$$\begin{aligned} f(X^0+h) - f(X^0) &= (x_0 + h_1)(y_0 + h_2) - x_0 y_0 \\ &= (y_0 h_1 + x_0 h_2) + h_1 h_2. \end{aligned}$$

Here $\|h\| = \sqrt{h_1^2 + h_2^2}$, $|h_1 h_2| \leqslant \|h\|^2 = (h)$, therefore

$$df(X^0, h) = y_0 h_1 + x_0 h_2,$$

and
$$y_0 = \frac{\partial f(X^0)}{\partial x}, \qquad x_0 = \frac{\partial f(X^0)}{\partial y}.$$

The converse of property 1° is, generally speaking, not true; the existence of all partial derivatives $\partial f(X)/\partial x_i$ $(i = 1, 2, \ldots, n)$ at a point $X^0(x_1^0, x_2^0, \ldots, x_n^0)$ or even in a certain neighbourhood of this point does not ensure the existence of the differential $df(X^0, h)$. However, the following proposition holds:

2°. If in some neighbourhood of the point $X^0(x_1^0, x_2^0, \ldots, x_n^0)$ there exist partial derivatives $\partial f(X)/\partial x_i$ $(i = 1, 2, \ldots, n)$ and they are continuous, then there exists a differential $df(X^0, h)$ at the point X^0.

(b) *The second definition of a differential of a function of n variables.* Consider the function $f(X) = f(x_1, x_2, \ldots, x_n)$ and the segment of a straight line $\{X^0 + th\}$ $(0 \leqslant t \leqslant 1)$ joining the points X^0 and $X^0 + h$, where $X^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ and $h = (h_1, h_2, \ldots, h_n)$, supposing that the whole of this segment lies inside the domain of definition of function f. In this straight line the function $f(X^0 + th)$ becomes a function of one variable t. Let us give the name of *differential* $df(X^0, h)$ of the function f at the point X^0, with the increment h, to the derivative

$$\overline{df(X^0, h)} = \frac{d}{dt} f(X^0 + th)\Big|_{t=0}, \tag{2.11}$$

if it exists for any h.

For a unit vector h, $\|h\| = 1$, this derivative

$$\frac{d}{dt} f(X^0 + th)\Big|_{t=0}$$

is called *the derivative of the function* $f(X)$ at the point X^0 in the direction of the vector h. It shows the rate of change of the function in the given direction. In particular, if h equals one of the unit vectors e_i $(i = 1, 2, \ldots, n)$ then

$$\overline{df(X^0, e_i)} = \frac{d}{dt} f(X^0 + te_i)\Big|_{t=0} = \frac{\partial f(X^0)}{\partial x_i},$$

i.e. partial derivatives are derivatives in the direction of the corresponding axes. Thus, as in the case of the definition (a), the existence of all partial derivatives follows from the existence of the differential. The converse conclusion does not hold.

EXAMPLE 2. Suppose the function $f(x, y)$ has the following form in polar coordinates in a plane:

$$f = \varrho \sin\left[\varphi\left(\frac{\pi}{2} - \varphi\right)(\pi - \varphi)\left(\frac{3\pi}{2} - \varphi\right)\frac{1}{\varrho}\right].$$

At the axes x and y, i.e. when $\varphi = 0, \pi/2, \pi, 3\pi/2$, we have $f \equiv 0$, therefore at the point $\Theta(0, 0)$ the partial derivatives $\partial f/\partial x = \partial f/\partial y = 0$; if, on the other hand, the vector h does not lie on any of the axes, the derivative $\frac{df(\Theta + th)}{dt}\Big|_{t=0} = \frac{df(th)}{dt}\Big|_{t=0}$ does not exist.

If in the neighbourhood of the point X^0 all derivatives $\partial f/\partial x_i$ $(i = 1, 2, \ldots, n)$ not only exist, but are continuous, then for any $h = (h_1, h_2, \ldots, h_n)$ there exist differentials $df(X^0, h) = \overline{df(X^0 h)}$, equal among themselves, i.e.

$$\left.\frac{df(X^0 + th)}{dt}\right|_{t=0} = \sum_{i=1}^{n} \frac{\partial f(X^0)}{\partial x_i} h_i. \tag{2.12}$$

If at the point X^0 there exists a differential $df(X^0, h)$ in the sense of definition (a) then there exists at that point also the differential $\overline{df(X^0, h)}$ equal to it. The converse proposition is, generally speaking, not true; the existence of $df(X^0, h)$ does not yet follow from the existence of the differential $\overline{df(X^0, h)}$ in the sense of the definition (b). This can be easily verified from the following examples.

Example 3. Let the function $f(X) = f(x, y) = \sqrt[3]{x^3 + y^3}$ be defined in a plane. Then, at the point $\Theta(0, 0)$ we have, for each vector $h = (h_1, h_2)$, $\Theta + th = th$ and

$$f(\Theta + th) = f(th) = t\sqrt[3]{h_1^3 + h_2^3},$$

whence

$$\frac{d}{dt} f(\Theta + th) = \sqrt[3]{h_1^3 + h_2^3},$$

in particular, for unit vectors $h = e_1(1, 0)$ and $h = e_2(0, 1)$ we obtain

$$\left.\frac{d}{dt} f(\Theta + te_1)\right|_{t=0} = \frac{\partial f(\Theta)}{\partial x} = 1,$$

$$\left.\frac{d}{dt} f(\Theta + te_2)\right|_{t=0} = \frac{\partial f(\Theta)}{\partial y} = 1.$$

Partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist over the whole plane, but they suffer a discontinuity at the point Θ. For example, for any $x \neq 0$, the derivative $[\partial f(x, y)]/\partial y = 3y^2(x^3 + y^3)^{-2/3}$, and for $y = 0$ we have

$$\frac{\partial f(x, 0)}{\partial y} = 0.$$

In this case, the expression

$$\overline{df(\Theta, h)} = \left.\frac{d}{dt} f(\Theta + th)\right|_{t=0} = \sqrt[3]{h_1^3 + h_2^3}$$

is not a linear function of h_1 and h_2 and, therefore, it cannot coincide with $df(\Theta, h)$ in the sense of definition (a). There is no such differential at the point Θ.

Example 4. Let us define the function $f(X) = f(x, y)$ in a plane as follows. Let $f = 0$ on the axis Oy and on the parabola q_1 (Fig. 7), whose equation is

$x = 2y^2$, and also over the whole region G_1 inside this parabola, i.e. for all points X for which $x > 2y^2$ (the shaded region in Fig. 7). On the parabola q, whose equation is $x = y^2$, $f(x, y) = \sqrt{x^2 + y^2}$. Finally, at every pair of segments AB and BC of the straight line $y = \text{const}$, where point A lies on the axis Oy, point B lies on the parabola q and point C on the parabola q_1, we shall regard $f(X)$ as a linear function of x. For negative x we put $f(x, y) = f(-x, y)$ and let $f(\Theta) = 0$ at the point $\Theta(0, 0)$.

For any h we have

$$\frac{d}{dt} f(\Theta + th)\bigg|_{t=0} = \frac{d}{dt} f(th)\bigg|_{t=0} = 0.$$

Indeed, if h lies on the axis Oy, then $f(th) \equiv 0$. If h lies outside Oy, then, for a certain $t_1 > 0$ dependent on h, the segment $\Theta + th$, when $0 \leqslant t < t_1$, lies

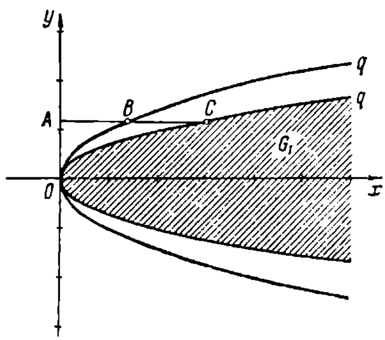

FIG. 7

wholly in the region G_1 or in the one symmetrical to it with respect to the axis Oy, where again $f(th) \equiv 0$, from the construction of the function. Thus, the function f has a differential in the sense of the definition (b) and

$$\overline{df(\Theta, h)} = 0.$$

Since $\overline{df}$ is identically equal to zero it can be regarded as a linear function of h_1 and h_2. Nevertheless, in the sense of the definition (a) the differential $df(\Theta, h)$ does not exist.

Indeed, if $df(\Theta, h)$ were to exist, the following equation would hold

$$df(\Theta, h) = \overline{df(\Theta, h)} = 0.$$

Since $f(\Theta) = 0$, it should be

$$f(h) = f(\Theta + h) = f(\Theta + h) - f(\Theta) = \varepsilon(\Theta) = o(\|h\|).$$

However, the latter is not true for h on the parabola q, because there $f(h) = \|h\|$. Thus, the differential $df(\Theta, h)$ does not exist, although $df(\Theta, h)$ is a linear function of h.

4. The name *gradient* of a function $f(X) = f(x_1, x_2, \dots, x_n)$ at the point X^0 is given to the vector with components $[\partial f(X^0)]/\partial x_i$:

$$\operatorname{grad} f(X^0) = \left(\frac{\partial f(X^0)}{\partial x_1}, \ \frac{\partial f(X^0)}{\partial x_2}, \ \dots, \ \frac{\partial f(X^0)}{\partial x_n}\right). \qquad (2.13)$$

Hence,

$$\|\operatorname{grad} f(X^0)\| = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f(X^0)}{\partial x_i}\right)^2}.$$

It follows from the definition of the gradient and the differential, that if the differential $df(X^0, h)$ exists, then

$$\left.\frac{df(X^0 + th)}{dt}\right|_{t=0} = \overline{df(X^0, h)} = (\operatorname{grad} f(X^0), h),$$

i.e. the differential $\overline{df}$ equals the scalar product of the gradient and the vector h. On the basis of Cauchy's inequality

$$\left|\left.\frac{df(X^0 + th)}{dt}\right|_{t=0}\right| \leqslant \|\operatorname{grad} f(X^0)\| \, \|h\|. \qquad (2.14)$$

We shall assume that $\operatorname{grad} f \neq 0$, i.e. $\sum_{i=1}^{n} (\partial f/\partial x_i)^2 \neq 0$. (2.14) is an equality when the vector h is collinear with the vector $\operatorname{grad} f$. In particular, if $h = h_0$ is a unit vector, i.e. $\|h_0\| = 1$, then

$$\frac{df(X^0 + th_0)}{dt} \leqslant \|\operatorname{grad} f(X^0)\|.$$

Let h_0 be a unit vector, whose direction coincides with the direction of the gradient: $h_0 = (1/\|\operatorname{grad} f\|) \operatorname{grad} f$. Then

$$\left.\frac{df(X^0 + th_0)}{dt}\right|_{t=0} = \|\operatorname{grad} f(X^0)\|. \qquad (2.15)$$

This means that the direction of the gradient is the direction along which the derivative is maximal, i.e. *the direction of the fastest increase of the function* $f(X)$ at the given point X^0.

The name of linear tangential manifold to the equipotential surface $f(X) = c$ at the point X^0 is given to an $(n-1)$-dimensional

manifold, $L_f(X^0)$, of vectors $g(g_1, g_2, \ldots, g_n)$ orthogonal to the vector of the gradient, i.e. satisfying the equation

$$df(X^0, g) = (\operatorname{grad} f, g) = \sum_{i=1}^{n} \frac{\partial f(X^0)}{\partial x_i} g_i = 0.$$

In the neighbourhood of the point X^0 every element X of the equipotential surface has the form

$$X = X^0 + g + \varepsilon \operatorname{grad} f, \tag{2.16}$$

where g is an element of the tangential manifold L and $\varepsilon = o(\|g\|)$. Conversely, when $\|g\|$ is sufficiently small, to every element $g \in L$ there corresponds an element X of the equipotential surface, represented in the form (2.16) with $\varepsilon = o(\|g\|)$.

The set of points of the form $X^0 + g$ with $g \in L$ forms a linear tangential hyperplane to the equipotential hyperplane $f(X) = c$ at the point X^0. In the sense of all that was said above, in the neighbourhood of the point X^0, the equipotential hyperplane and the tangential hyperplane coincide to the first order of small quantities. The direction of the gradient is the direction of the normal to the equipotential hyperplane at the point X^0.

The name *orthogonal trajectory* of a system of equipotential surface is given to the curve $X = X(t)$ satisfying the differential equation

$$\frac{dX}{dt} = \operatorname{grad} f(X)$$

or, in the coordinate form, the set of equations

$$\frac{dx_i}{dt} = \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i} \quad (i = 1, 2, \ldots, n).$$

If at every point of the set $\mathfrak{M}$ (region, closed region) $\operatorname{grad} f \neq 0$, then, through every point of $\mathfrak{M}$ there passes one, and only one, orthogonal trajectory. The tangent to the orthogonal trajectory at its every point has the direction of the gradient at that point.

§ 2. Derivatives and Differentials of Higher Orders. Taylor's Series

1. Suppose, the function $f(X) = f(x_1, x_2, \ldots, x_n)$ and its partial derivative $U_i(X) = \partial f(x)/\partial x_i$ are defined at every point of the region G. If the function U_i, in its turn, has a partial derivative $\partial U_i/\partial x_j$ at every point of G, it is said that at every point of the region G there is defined a *second partial derivative*

$$\frac{\partial^2 f(X)}{\partial x_j\, \partial x_i} = \frac{\partial U_i}{\partial x_j},$$

or *a partial derivative of the second order.*

The partial derivatives of mth order are defined in a recurrent fashion. Suppose, for the function $f(X) = f(x_1, x_2, \ldots, x_n)$ in the region G, there are defined derivatives of the $(m-1)$th order and there exists a derivative

$$U(x_1, x_2, \ldots, x_n) = \frac{\partial^{m-1} f(x_1, x_2, \ldots, x_n)}{\partial x_{i_2}\, \partial x_{i_3} \ldots \partial x_{i_m}}.$$

Then, if there exists in G a derivative $\partial U/\partial x_i$, it is called *an m-th partial derivative*

$$\frac{\partial^m f(x_1, x_2, \ldots, x_n)}{\partial x_{i_1}\, \partial x_{i_2} \ldots \partial x_{i_m}} = \frac{\partial U}{\partial x_{i_1}}$$

of the function f. It is obtained by means of the successive differentiation of the function f with respect to the variables $x_{i_m}, x_{i_{m-1}}, \ldots, x_{i_2}, x_{i_1}$.

2. The partial derivatives of higher orders can be determined with the help of difference operators.

The product of operators $\Delta_{h_k}^{x_k}\Delta_{h_{k-1}}^{x_{k-1}}, \ldots, \Delta_{h_2}^{x_2}\Delta_{h_1}^{x_1}$ is an operator consisting of the consecutive application of the operators $\Delta_{h_1}^{x_1}, \Delta_{h_2}^{x_2}, \ldots, \Delta_{h_k}^{x_k}$. For example, for the function $f(x, y)$ we have

$$\Delta_{h_2}^{y}\Delta_{h_1}^{x} f(x, y) = \Delta_{h_2}^{y}(\Delta_{h_1}^{x} f(x, y) = \Delta_{h_2}^{y}[f(x + h_1, y) - f(x, y)]$$

$$= [f(x + h_1, y + h_2) - f(x, y + h_2)] - [f(x + h_1, y) - f(x, y)]$$

$$= f(x + h_1, y + h_2) - f(x, y + h_2) - f(x + h_1, y) + f(x, y).$$

Similarly

$$\Delta^x_{h_1}\Delta f^y_{h_2}(x, y) = \Delta^x_{h_1}(\Delta^y_{h_2}f(x, y)) = \Delta^x_{h_1}[f(x, y + h_2) - f(x, y)]$$

$$= [f(x + h_1, y + h_2) - f(x + h_1, y)] - [f(x, y + h_2) - f(x, y)]$$

$$= f(x + h_1, y + h_2) - f(x + h_1, y) - f(x, y + h_2) + f(x, y).$$

It follows hence that

$$\Delta^y_{h_2}\Delta^x_{h_1}f(x, y) = \Delta^x_{h_1}\Delta^y_{h_2}f(x, y),$$

i.e. the operators $\Delta^x_{h_1}$ and $\Delta^y_{h_2}$ are commutative. In general, operators $\Delta^{x_i}_{h_i}$ and $\Delta^{x_j}_{h_j}$ are commutative. The product of k operators $\Delta^{x_i}_{h_i}$ $(i = 1, 2, \ldots, k)$ is the difference operator of the kth order.

The name *differential derivative of the k-th order*

$$\overline{\frac{\partial^k f(x_1, x_2, \ldots, x_n)}{\partial x_{i_1}^{k_1}\, \partial x_{i_2}^{k_2} \ldots \partial x_{i_m}^{k_m}}} \quad (k_1 + k_2 + \cdots + k_m = k)$$

of the function $f(X) = f(x_1, x_2, \ldots, x_n)$ at the point $X(x_1, x_2, \ldots, x_n)$ is given to the limit (if it exists) of the ratio

$$\overline{\frac{\partial^k f(x_1, x_2, \ldots, x_n)}{\partial x_{i_1}^{k_1}\, \partial x_{i_2}^{k_2} \ldots \partial x_{i_m}^{k_m}}} = \lim_{\substack{h_1 \to 0,\\ h_2 \to 0,\ \ldots,\ h_m \to 0}} \times$$

$$\times \frac{\left(\Delta^{x_{i_1}}_{h_1}\right)^{k_1} \left(\Delta^{x_{i_2}}_{h_2}\right)^{k_2} \ldots \left(\Delta^{x_{i_n}}_{hn}\right)^{k_m} f(x_1, x_2, \ldots, x_n)}{h_1^{k_1} h_2^{k_2} \ldots h_m^{k_m}}. \tag{2.17}$$

It follows from the commutability of the operators $\Delta^{x_{ij}}_{h_j}$ that in the symbol of differentiation on the left side of (2.17) the order of the symbols $\partial x_{i_1}, \partial x_{i_2}, \ldots, \partial x_{i_m}$ can be altered in any way desired, for example

$$\overline{\frac{\partial^2 f(x, y)}{\partial x\, \partial y}} = \overline{\frac{\partial^2 f(x, y)}{\partial y\, \partial x}}.$$

If there exists in the region G a continuous, kth successive, derivative in the sense of the definition in sec. 1,

$$\frac{\partial^k f(x_1, x_2, \ldots, x_n)}{\partial x_{i_1}^{k_1}\, \partial x_{i_2}^{k_2} \ldots \partial x_{i_m}^{k_m}}, \tag{2.18}$$

then there also exists the kth differential derivative

$$\overline{\frac{\partial^k f(x_1, x_2, \ldots, x_n)}{\partial x_{i_1}^{k_1} \, \partial x_{i_2}^{k_2} \ldots \partial x_{i_m}^{k_m}}}$$

coincident with it. Given these conditions, there exist, as well as the derivative (2.18), all the kth derivatives, equal to it, differing from (2.18) only in the order in which the operations of differentiation $\partial/\partial x_{i_1}$, $\partial/\partial x_i, \ldots$ are carried out. For example, if $\partial^2 f(x, y)/\partial x \, \partial y$ is continuous, then there exist the following derivatives, which coincide with it:

$$\overline{\frac{\partial^2 f(x, y)}{\partial x \, \partial y}} = \overline{\frac{\partial^2 f(x, y)}{\partial y \, \partial x}} = \frac{\partial^2 f(x, y)}{\partial y \, \partial x} = \frac{\partial^2 f(x, y)}{\partial x \, \partial y}.$$

In this sense one speaks of the commutativity of operations of partial differentiations $\partial/\partial x$ and $\partial/\partial y$ applied to the function $f(x, y)$ and, in general, about the commutativity of operations of partial differentiation $\partial/\partial x_i$, $\partial/\partial x_j$ $(i, j = 1, 2, \ldots, n)$ applied to the function $f(x_1, x_2, \ldots, x_n)$. The convention is to denote by the symbol

$$\frac{\partial^k f(x_1, x_2, \ldots, x_n)}{\partial x_1^{k_1} \, \partial x_2^{k_2} \ldots \partial x_n^{k_n}} \tag{2.19}$$

any derivative of the order $k = k_1 + k_2 + \cdots + k_n$ (where some k_i may equal zero) obtained by the k_1-fold application of differentiation with respect to x_1, the k_2-fold differentiation with respect to $x_2, \ldots$, the k_n-fold differentiation with respect to x_n, independent of the order in which this differentiation was carried out. For example, the symbol

$$\frac{\partial^3 f(x, y)}{\partial x^2 \, \partial y}$$

denotes any of the derivatives

$$\frac{\partial^3 f(x, y)}{\partial x \, \partial x \, \partial y} = \frac{\partial^3 f(x, y)}{\partial x \, \partial y \, \partial x} = \frac{\partial^3 f(x, y)}{\partial y \, \partial x \, \partial x}$$

equal among themselves.

The function $f(X) = f(x_1, x_2, \ldots, x_n)$, defined in the region G, is called *a function of the class* $c_n = C_{n,G}$, if it has all the consecutive derivatives of the first n orders, which are continuous in G. These

derivatives, firstly, coincide with the corresponding differential derivatives, and, secondly, the derivatives of the kth order, for $2 \leqslant k \leqslant n$, which differ only in the sequence of application of operators of partial differentiation, coincide.

3. As was said before, the operators of partial differentiation $D_i = \partial/\partial x_i$ map the function f of the class C_1 into the function

$$D_i f = \frac{df}{\partial x_i}$$

of the class C.

The operator $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ (among $i_1, \ldots, i_k$ there may be equal ones) is an operator from $C_{k,G}$ into C for any function $f \in C_k$:

$$D_{i_1} D_{i_2} \ldots D_{i_k} f = \frac{\partial^k f}{\partial x_{i_1}\, \partial x_{i_2} \ldots \partial x_{i_k}}.$$

For the function $f \in C_2$, the operators D_i and D_j are commutative:

$$D_i D_j f = D_j D_i f$$

(see sec. 2).

Using this notation, we write down the differential in the form

$$df(X, h) = \left(\sum_{i=1}^{n} h_i D_i\right) f(X).$$

Then, we put, for the function $f \in C_2$, by definition

$$d^2 f(X, h) = d\,\{d[f(X, h)]\} = \left(\sum_{i=1}^{n} h_i D_i\right)^2 f(X) \tag{2.20}$$

and in general, for $f \in C_k$

$$d^k f(X, h) = \left(\sum_{i=1}^{n} h_i D_i\right)^k f(X). \tag{2.21}$$

For example, for $h = (h_1, h_2)$ we have, for a function of two variables, $f(x, y)$:

$$\begin{aligned} d^k f(X, h) &= (h_1 D_x + h_2 D_y)^k f(x, y) = \sum_{m=0}^{k} C_k^m h_1^m h_2^{k-m} D_k^m D_y^{k-m} f(x, y) \\ &= \sum_{m=0}^{k} C_k^m \frac{\partial^k f(x, y)}{\partial x^m\, \partial y^{k-m}}\, h_1^m h_2^{k-m}. \end{aligned} \tag{2.22}$$

The formula (2.11) can be generalized

$$\overline{d^k f(X, h)} = \left.\frac{d^k}{dt^k} f(X + th)\right|_{t=0}. \tag{2.23}$$

Taylor's formula

4. Just as in the case of a function of one variable, the purpose of Taylor's formula for a function of n variables is to represent this function approximately by means of a polynomial.

THEOREM 1. *Every polynomial $P(X)$ of k-th degree of n variables $x_1, x_2, \ldots, x_n$,*

$$
\begin{aligned}
P(X) &= P(x_1, x_2, \ldots, x_n) \\
&= a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i_1, i_2 = 1}^{n} a_{i_1 i_2} x_{i_1} x_{i_2} + \cdots \\
&\quad + \sum_{i_1, i_2, \ldots, i_k = 1}^{n} a_{i_1} a_{i_2} \ldots {}_{i_k} x_{i_1} x_{i_2} \ldots x_{i_k},
\end{aligned}
$$

satisfies the identity

$$
\begin{aligned}
&P(x_1 + h_1, x_2 + h_2, \ldots, x_n + h_n) = P(x_1, x_2, \ldots, x_n) \\
&+ \sum_{s=1}^{k} \frac{1}{s!} (h_1 D_1 + h_2 D_2 + \cdots + h_n D_n)^s P(x_1, x_2, \ldots, x_n), \quad (2.24)
\end{aligned}
$$

or, for short:

$$
\begin{aligned}
P(X + h) &= P(X) + \sum_{s=1}^{k} \frac{1}{s!} (h_1 D_1 + \cdots + h_n D_n)^s P(X) \\
&= P(X) + dP(X, h) + \frac{1}{2!} d^2 P(X, h) + \cdots + \frac{1}{k!} d^k P(X, h).
\end{aligned}
\quad (2.25)
$$

THEOREM 2. *Let G be a region of E_n, $F(X)$ be a function in the class $C_{k,G}$ and $X \in G$. Then, for $X + h \in G$,*

$$
F(X + h) - F(X) = \sum_{s=1}^{k} \frac{1}{s!} \left(\sum_{i=1}^{n} h_i D_i \right)^s F(X) + \varepsilon(h), \quad (2.26)
$$

where $\varepsilon(h) = o(\|h\|^k)$.

The sum in the right-hand side of (2.26) represents a polynomial of the kth degree with respect to the coordinates $h_1, h_2, \ldots, h_n$ of the vector h.

The converse also holds:

THEOREM 3. *If $f(X + h) = P_k(h) + \varepsilon(h)$, where $P_k(h)$ is a poly-*

nomial of the k-th degree with respect to $h_1, h_2, \ldots, h_n$ *and* $\varepsilon(h) = o(\|h\|^k)$, *then*

$$P_k(h) = f(X) + \sum_{s=1}^{k} \frac{1}{s!} (h_1 D_1 + \cdots + h_n D_n)^s f(X). \quad (2.27)$$

Theorem 2 can be made more exact as follows:

THEOREM 4. *If* $f(X) \in C_{k+1,G}$ *and* $X \in G$, *then for* $X + h \in G$

$$f(X + h) = f(X) + \sum_{s=1}^{k} \frac{1}{s!} \left(\sum_{i=1}^{n} h_i D_i \right)^s f(X) + \varepsilon(h), \quad (2.28)$$

where $\varepsilon(h) = o(\|h\|^{k+1})$.

If $f(X)$ has all derivatives of any order in the neighbourhood of the point X^0, then it is possible to define formally a power series with respect to the powers of $h_1, h_2, \ldots, h_n$ – the coordinates of vector h:

$$f(X^0) + (h_1 D_1 + h_2 D_2 + \cdots + h_n D_n) f(X^0) + \cdots$$
$$+ \frac{1}{k!} (h_1 D_1 + \cdots + h_n D_n)^k f(X^0) + \cdots \quad (2.29)$$

The series (2.29) is called *Taylor's series* for $f(X)$. For example, for a function of two variables $f(X) = f(x, y)$ this series has the form

$$f(x_0, y_0) + \left(\frac{\partial f(x_0, y_0)}{\partial x} h_1 + \frac{\partial f(x_0, y_0)}{\partial y} h_2 \right) + \cdots$$
$$+ \frac{1}{k!} \sum_{s=0}^{k} C_k^s \frac{\partial^k f(x_0, y_0)}{\partial x^s \, \partial y^{k-s}} h_1^s h_2^{k-s} + \cdots. \quad (2.30)$$

Limiting ourselves to the first m terms of the series (2.29) we have

$$f(X^0 + h) = f(X^0) + \sum_{s=1}^{m} \frac{1}{s!} \left(\sum_{i=1}^{n} h_i D_i \right)^s f(X^0) + r_n(X, h), \quad (2.31)$$

where $r_n(X, h)$ *is the remainder:*

$$r_n = \frac{1}{(n+1)!} \left(\sum_{i=1}^{n} h_i D_i \right)^{n+1} f(X^0 + \theta h), \quad 0 < \theta < 1.$$

If, for a given X^0 and $\|h\| < \varrho$, there takes place $r_n(X, h) \to 0$, for $n \to \infty$, then, for $X = X^0 + h$, $\|h\| < \varrho$, the function $f(X)$ can

be represented by Taylor's series

$$f(X) = f(X^0 + h) = f(X^0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{i=0}^{n} h_i D_i \right)^k f(X^0). \tag{2.32}$$

Comparing the right-hand side of (2.32) with the expansion of function e^t, it is easy to note that it may be written symbolically in the form

$$e^{\sum_{i=1}^{n} h_i D_i} f(X^0).$$

Owing to this, the equation (2.32) can be written down thus:

$$\begin{aligned} f(X^0 + h) &= f(x_1^0 + h_1, x_2^0 + h_2, \ldots, x_n^0 + h_n) \\ &= e^{h_1 D_1 + \cdots + h_n D_n} f(x_1^0, x_2^0, \ldots, x_n^0). \end{aligned} \tag{2.33}$$

§ 3. Polynomials of Differential Operators

1. Let $P_k(t) = P_k(t_1, t_2, \ldots, t_n)$ be a polynomial of kth degree with respect to $t_1, t_2, \ldots, t_n$:

$$P_k(t) = \sum_{s=0}^{k} \left(\sum_{k_1 + k_2 + \cdots + k_n = s} a_{k_1 k_2 \ldots k_n} t_1^{k_1} t_2^{k_2} \ldots t_n^{k_n} \right)$$

and $D_i = \partial/\partial x_i$ the operator of partial differentiation. The polynomial $P_k(D)$ of operators D_i is understood as the operator from C_k to C, determined by the equation

$$\begin{aligned} P_k(D) &= P_k(D_1, D_2, \ldots, D_n) \\ &= \sum_{s=0}^{k} \left(\sum_{k_1 + k_2 + \cdots + k_n = s} a_{k_1 k_2 \ldots k_n} D_1^{k_1} D_2^{k_2} \ldots D_n^{k_n} \right). \end{aligned} \tag{2.34}$$

For example, to the polynomial $P_5(t) = 3t_1^3 t_2^2 + t_1^4 t$, there corresponds the operator

$$P_5(D) = 3D_1^3 D_2^2 + D_1^4 D_2.$$

Let us introduce the following notation for the derivatives of the order $m < k$ of the polynomial $P_k(t)$:

$$P_{k, r_1, r_2, \ldots, r_n}(t) = \frac{\partial^m P_k(t)}{\partial t_1^{r_1} \partial t_2^{r_2} \ldots \partial t_n^{r_n}} \quad (r_1 + \cdots + r_n = m). \tag{2.35}$$

We arrive at a new polynomial of an operator, i.e. at a new operator from C_{k-m} into C, denoted by the symbol $P_{k, r_1, r_2, \ldots, r_n}(D)$ or $P_{r_1, r_2, \ldots, r_n}(D)$ (the original index k, showing the degree of the initial polynomial, i.e. the order of the initial operator, can be dropped).

Thus, for the polynomial $P(t) = 3t_1^3t_2^2 + t_1^4t_2$ we obtain

$$P_{2,1}(t) = 36t_1t_2 + 12t_1^2,$$

$$P_{1,2}(t) = 18t_1^2.$$

It follows that

$$P_{2,1}(D) = 36D_1D_2 + 12D_1^2 = 36\frac{\partial^2}{\partial x_1\,\partial x_2} + 12\frac{\partial^2}{\partial x_1^2},$$

$$P_{1,2}(D) = 18D_1^2 = 18\frac{\partial^2}{\partial x_1^2}.$$

If $P(t)$ is a polynomial of kth degree, $Q(t)$ is a polynomial of degree l and

$$R(t) = P(t)Q(t),$$

then

$$R(D) = P(D)Q(D) = Q(D)P(D).$$

For example, to the polynomial $t_1^4 - t_2^4 = (t_1^2 + t_2^2)(t_1^2 - t_2^2)$ there corresponds the operator

$$\frac{\partial^4}{\partial x_1^4} - \frac{\partial^4}{\partial x_2^4} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right).$$

For *Laplace's operator*

$$\Delta = \sum_{i=1}^{n} D_i^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \tag{2.36}$$

$$\Delta^2 = \left(\sum_{i=1}^{n} D_i^2\right)^2 = \sum_{i=1}^{n} D_i^4 + 2\sum_{\substack{i,j=1\\ i>j}}^{n}{}' D_i^2D_j^2, \tag{2.37}$$

where the sign $'$ at the sum means, that the summing is not carried out in all values of i and j, but only in those that satisfy the condition $i > j$, or

$$\Delta^2 = \sum_{i=1}^{n} \frac{\partial^4}{\partial x_i^4} + 2\sum_{\substack{i,j=1\\ n>j}}^{n}{}' \frac{\partial^4}{\partial x_i^2\,\partial x_j^2} \tag{2.38}$$

(biharmonic operator).

For a polynomial of an operator there is the following *generalization of Leibniz's formula.* Let $P(t) = P(t_1, t_2, \ldots, t_n)$ be a polynomial of kth degree with respect to t_i $(i = 1, 2, \ldots, n)$, then

$$P(D)(uv) = \sum_{r=0}^{k} \sum_{r_1+r_2+\cdots+r_n=r} P_{r_1,r_2,\ldots,r_n}(D)\, u \frac{D_1^{r_1} D_2^{r_2} \ldots D_n^{r_n} v}{r_1!\, r_2! \ldots r_n!}. \tag{2.39}$$

§ 4. The Differentiation of Mappings from E_n into E_m

1. If to every vector $X(x_1, x_2, \ldots, x_n)$ of the space E_n there corresponds a definite vector $Y(y_1, y_2, \ldots, y_m)$ of space E_m, it is said that the *operator* or *image* from E_n into E_m has been defined. This is written down in the form

$$Y = f(X) \tag{2.40}$$

or in the coordinate form

$$y_i = f_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, m). \tag{2.41}$$

Thus, if the operator (2.40) from E_n into E_m is given, this is equivalent to being given a system (2.41) of m functions of n variables. Operator $f(X)$ is called *continuous*, if all functions defining it are continuous.

2. By analogy to a differential of a function, it is possible to define *the differential of the operator* F. Let X^0 and $X^0 + H$, where $H = (h_1, h_2, \ldots, h_n)$, be vectors in E_n. The difference

$$f(X^0 + H) - f(X^0), \tag{2.42}$$

the "increment in value" of the operator f, is an m-dimensional vector, whose components are

$$f_i(X^0 + H) - f_i(X^0)$$
$$= f_i(x_1^0 + h_1, x_2^0 + h_2, \ldots, x_n^0 + h_n) - f_i(x_1^0, x_2^0, \ldots, x_n^0)$$
$$(i = 1, 2, \ldots, m). \tag{2.43}$$

Suppose, the increment (2.42) can be represented in the form

$$f(X^0 + H) - f(X^0) = AH + \varepsilon(H), \tag{2.44}$$

where A is the linear operator from E_n into E_m and

$$\|\varepsilon(H)\| = o(\|H\|). \tag{2.45}$$

Then AH—the linear operator with respect to H—"the principal part" of the increment is called *the differential of the operator f* at the point $X^0 \in E_n$ and is denoted

$$df(X^0, H) = AH. \tag{2.46}$$

The operator f is said to be *differentiable at the point* X^0. For a differentiable operator

$$f(X^0 + H) - f(X^0) = df(X^0, H) + \varepsilon(H),$$

and $\|\varepsilon(H)\| = o(\|H\|)$.

The components of the linear operator AH are linear functions of $H = (h_1, h_2, \ldots, h_n)$:

$$A_i H = df_i(X^0, H) = \sum_{j=1}^{n} \frac{\partial f_i(X^0)}{\partial x_j} h_j \quad (i = 1, 2, \ldots, m). \tag{2.47}$$

Thus, the linear operator $AH = df(X^0, H)$ is defined by the matrix

$$\|A\| = \left(\frac{\partial f_i(X^0)}{\partial x_j}\right)_{j=1,2,\ldots,n}^{i=1,2,\ldots,m}, \tag{2.48}$$

of m rows and n columns. This matrix is called a *Jacobi's matrix.*

If we denote Jacobi's matrix of the operator f by $f'(X^0)$, the expression for the differential of the operator can be written down in the form

$$df(X^0, H) = f'(X^0)H, \tag{2.49}$$

which represents an analogue of the connection between the differential and the derivative of a function of one variable. Thus, Jacobi's matrix of an operator is the analogue of the derivative. Further analogues will be considered in the next part of this book.

In the particular case of mapping E_1 into E_2, i.e. when $m = n = 1$, the operator f is reduced to the function $f(x)$ of one variable and its Jacobi matrix is reduced to the scalar df/dx. If $m = 1$, $n > 1$, the operator f is reduced to a function of n variables. Its differential is then a differential of this function. The Jacobi matrix in this case becomes a row matrix

$$\left\| \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\|.$$

When $m > 1$, $n = 1$, the operator f is reduced to the vector function of a scalar argument. Its Jacobi matrix is a column matrix

$$\left\| \begin{matrix} \dfrac{df_1}{dx} \\ \dfrac{df_2}{dx} \\ \cdots \\ \dfrac{df_m}{dx} \end{matrix} \right\|.$$

3. When X and H are fixed, the operator f on the straight line $X + tH$ is a vector function of a scalar argument t. We have

$$df(X, H) = \frac{d}{dt} f(X + tH)\Big|_{t=0}$$

(compare formula (1.20)).

It is said that the operator $f = (f_1, f_2, \ldots, f_m)$ is *an operator of the class* C_k, if functions f_i $(i = 1, 2, \ldots, m)$ belong the class C_k (see § 2). If $f \in C_k$, then, for the operator f, it is possible to determine the differential of the kth order:

$$d^k f(X, H) = \frac{d^k}{dt^k} f(X, tH)\Big|_{t=0}.$$

The differential $d^k f(X, H)$ can be considered as an operator of vector H. In this case the components of the operator $d^k f(X, H)$ are the function $d^k f_i(X, H)$ $(i = 1, 2, \ldots, n)$.

§ 5. Extrema

1. Suppose, the function $f(X)$ is defined in some set of the space E_n.

If there is, among the values of the function, one that is the *greatest*, it is called the *absolute maximum* of the function, and the point at which the maximum is reached is called *the point of absolute maximum*. Similarly, *the absolute minimum* is defined as *the least value of a function* (in the region of its definition). *The point of*

absolute minimum is the point at which the function takes the least possible value. The absolute minimum and the absolute maximum are called *absolute extrema* and the points at which the function takes these values are called *points of absolute extrema*.

The absolute maximum is the upper bound of the values of the function and the absolute minimum is their lower bound in the region of its definition. Therefore, only bounded functions can have an absolute maximum, i.e. the boundedness of the function is a necessary condition of the existence of an absolute extremum. However, the boundedness of the function is an insufficient condition of the existence of an extremum. Thus, for example, the function $z = 1/[(x^2 + 1)(y^2 + 1)]$ is bounded from below, because $z > 0$, and $\lim_{x\to\infty} z = 0$ and $\lim_{y\to\infty} z = 0$; it means that 0 is the lower bound of the function, but 0 is not the value of the function, and so there is no smallest value among all the values of the function.

The most general *conditions of the existence of an absolute extremum* are given by

THEOREM 5 *(Weierstrass). A function, continuous in a bounded closed set, has a greatest and a smallest value.*

Hence follow the following theorems:

THEOREM 6. *If a continuous function is defined in all space and when* $\|X\| \to \infty$ *we have* $\lim f(X) = +\infty$ ($\lim f(X) = -\infty$), *the function reaches an absolute minimum (maximum).*

THEOREM 7. *If a function is continuous in an unbounded closed set and when* $\|X\| \to \infty$ *we have* $f(X) \to +\infty$ ($f(X) \to -\infty$), *the function reaches an absolute minimum (maximum).*

THEOREM 8. *If the function* $y = f(X)$ *is continuous in the bounded set* E, *containing a point* X^0, *such, that for every boundary point of set* E, *there can be found a neighbourhood in which* $f(X) > f(X^0)$ ($f(X) < f(X^0)$), *then the function* $f(X)$ *in* E *reaches an absolute minimum (maximum).*

Using these theorems, it is possible in many cases to prove the existence of an extremum of a function.

An absolute extremum of a function may be reached at inner as well as at boundary points of a region. Methods of finding inner extremal points only are discussed below. The finding of boundary extremal points is fraught with great difficulties.†

† Certain cases of finding such extrema comprise one of the basic problems in linear programming.

2. Suppose the function $y = f(X)$ is defined in the open set G. If we can find a neighbourhood for the point X^0, at all points X of which the following inequality holds,

$$f(X^0) > f(X) \quad (X \neq X^0), \tag{2.50}$$

then X^0 is a point of *strict relative maximum*. If in some neighbourhood of the point X^0, the following inequality holds,

$$f(X^0) < f(X) \quad (X \neq X^0), \tag{2.51}$$

then X^0 is a point of *strict relative minimum*. Points of relative maximum and minimum are called *points of relative extremum*. If in a certain neighbourhood of the point X^0 the following inequality holds,

$$f(X^0) \geqslant f(X) \quad (f(X^0) \leqslant f(X)),$$

then X^0 is a point of *non-strict relative extremum*.

From the definitions quoted, it follows that in some neighbourhood of the point of relative extremum the increment of the function $\Delta f = f(X) - f(X^0)$ preserves its sign.

Points of relative extremum can be found sometimes by means of artificial techniques. There are general methods of finding extremal points for differentiable functions with the help of necessary and sufficient conditions.†

3. The conditions, necessary for an extremum are given below.

THEOREM 9. *If at the point of relative extremum, X^0, the first differential exists, it is identically equal zero, i.e.*

$$df(X^0) \equiv \sum_{i=1}^{n} \frac{\partial f(X^0)}{\partial x_i}\, dx_i \equiv 0. \tag{2.52}$$

Points X^0 at which the first differential does not exist may also be points of relative extrema.

The necessary condition means that if the function is differentiable at the point of relative extremum, then all its derivatives of the first order are equal zero:

$$\frac{\partial f(X^0)}{\partial x_i} = 0 \quad (i = 1, 2, \ldots, n). \tag{2.53}$$

In general, points satisfying conditions (2.53) are called *stationary*.

† Necessary and sufficient conditions for an extremum of a function of one variable are discussed in Chapter I, § 3.

THEOREM 10. *If a function is continuously differentiable twice at the point of relative maximum (minimum), the second differential is a non-positive (non-negative) form.*

The second differential of the function

$$d^2f(X^0) = \sum \frac{\partial^2 f(X^0)}{\partial x_i\, \partial x_k}\, dx_i\, dx_k \tag{2.54}$$

is a quadratic form of the differentials of the arguments. Non-positiveness (non-negativeness) of the quadratic form means that it cannot be positive (negative) for any values of variables.

Theorem 9 establishes the necessary conditions for an extremum only.

4. The sufficient conditions of an extremum are based on the representation of a twice continuously differentiable function in the neighbourhood of a stationary point X^0 in the form

$$\begin{aligned} f(X^0 + h) &= f(x_1^0 + h_1, x_2^0 + h_2, \ldots, x_n^0 + h_n) \\ &= f(X_0) + \tfrac{1}{2}\, d^2 f(X^0, h) + o(\|h\|^2), \end{aligned} \tag{2.55}$$

where

$$\tfrac{1}{2}\, d^2 f(X^0, h) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f(X^0)}{\partial x_i\, \partial x_j}\, h_i h_j. \tag{2.56}$$

It follows hence, that in a sufficiently small neighbourhood of the point X^0 the sign of the difference $f(X^0 + h) - f(X^0)$ coincides with the sign of the second differential, i.e. with the sign of the quadratic form (if it is of constant sign):

$$\sum_{i,j=1}^{n} \frac{\partial^2 f(X^0)}{\partial x_i\, \partial x_j}\, h_i h_j.$$

THEOREM 11. *If the function $y = f(X)$ is continuously differentiable twice in the stationary point P_0 and the second differential is a negative definite form, i.e.*

$$d^2 f(X^0) = \sum_{i,j=1}^{n} \frac{\partial^2 f(X^0)}{\partial x_i\, \partial x_j}\, h_i h_k < 0, \tag{2.57}$$

then the point X^0 is a point of relative maximum.

THEOREM 12. *If the function $y = f(X)$ is continuously differentiable twice at the stationary point X^0, and the second differential at this point is a positive definite form, i.e.*

$$d^2f(X^0) = \sum \frac{\partial^2 f(X^0)}{\partial x_i \, \partial x_j} h_i h_k > 0. \tag{2.58}$$

then X^0 is a point of relative minimum.

In order to investigate the sign of the quadratic form, the following theorem may be used.

THEOREM 13 *(Sylvester). In order that the quadratic form*

$$I_n = \sum_{i,k=1}^{n} a_{ik} x_i x_k \tag{2.59}$$

be positive definite, it is necessary and sufficient that all principal minors of its discriminant

$$\Delta_1 = a_{11}, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \ldots, \quad \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \tag{2.60}$$

be positive.

We now show how to apply the conditions for an extremum to the investigation of a function of two variables $z = f(x, y)$. To find the stationary points, we solve the simultaneous equations

$$z_x = 0, \quad z_y = 0. \tag{2.61}$$

Then we calculate for each stationary point X^0 the coefficients of the second differential

$$a_{11} = \frac{\partial^2 f(X^0)}{\partial x^2}, \quad a_{12} = \frac{\partial^2 f(X^0)}{\partial x \, \partial y}, \quad a_{22} = \frac{\partial^2 f(X^0)}{\partial y^2},$$

we compile the discriminant $a_{11}a_{22} - a_{12}^2$ and we determine its sign. If $a_{11}a_{22} - a_{12}^2 > 0$ then X^0 is an extremal point, and for $a_{11} > 0$ the point X^0 is a minimal point, but for $a_{11} < 0$ the point X^0 is a maximal point. (When the discriminant is positive a_{11} and a_{22} are of similar sign.) If $a_{11}a_{22} - a_{12}^2 < 0$, there is no extremum at point X^0, since the second necessary condition for the existence of an extremum does not hold. If $a_{11}a_{22} - a_{12}^2 = 0$, then in order to investigate the point X^0 for an extremum, it is necessary to consider the differential of the third order.

The absolute extremum and the relative extremum are called *unconditional extrema.* In solving problems involving an unconditional extremum, the function $y = f(x_1, x_2, \ldots, x_n)$ of n independent variables is considered.

5. Suppose, the function $y = f(X) = f(x_1, \ldots, x_n)$ is defined in an $(n-k)$-dimensional manifold E, given by the set of equations

$$\varphi_i(x_1, x_2, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, k). \tag{2.62}$$

The point X^0 of the manifold E is called *the point of conditional relative maximum (minimum),* if there can be found a neighbourhood of X^0 in the manifold E, such that the value of the function at the point X is greatest (smallest) in this neighbourhood. The definition of the conditional relative extremum is similar to the definition of the conditional extremum, except that, instead of taking the neighbourhood of the extremal point of a function of n variables in an n-dimensional Euclidean space R_n, it is taken in the given manifold E (i.e. we consider a set of points of E close to the point X^0).

The point X^0 of the manifold E is called *the point of conditional absolute maximum (minimum),* if the value of the function at this point of $f(X^0)$ is its greatest (least) value in the manifold E. To find points of conditional extremum it is usual to employ the following *rule of Lagrange's factors:* let us construct an auxiliary function F with the aid of the given function $f(X)$, the equations (2.62) and auxiliary factors *(Lagrange factors)* λ_i:

$$F = f + \sum \lambda_i \varphi_i. \tag{2.63}$$

We find partial derivatives $\partial F/\partial x_m$ $(m = 1, 2, \ldots, n)$, make them equal zero and add to this set of n equations the set of k equations of constraint. We obtain a set of $n + k$ simultaneous equations:

$$\left.\begin{aligned} \frac{\partial f}{\partial x_m} + \sum_{i=1}^{n} \lambda_i \frac{\partial \varphi_i}{\partial x_m} &= 0 \quad (m = 1, 2, \ldots, n), \\ \varphi_i(x_1, x_2, \ldots, x_n) &= 0 \quad (i = 1, 2, \ldots, n). \end{aligned}\right\} \tag{2.64}$$

Having solved it, we find the coordinates of all *conditional-stationary* points among which the required points of conditional extremum can be found. This rule may be used in the case, when the given

functions f and φ_i are differentiable, and the manifold is $(n-k)$-dimensional, i.e. if the matrix

$$\left\| \begin{matrix} \dfrac{\partial \varphi_1}{\partial x_1} \cdots \dfrac{\partial \varphi_1}{\partial x_n} \\ \cdots\cdots\cdots \\ \dfrac{\partial \varphi_k}{\partial x_1} \cdots \dfrac{\partial \varphi_k}{\partial x_n} \end{matrix} \right\|$$

is of the rank k. In these conditions the set (2.64) determines Lagrange's factors uniquely, therefore also the function F *(Lagrange's function)* is determined uniquely. The problem of investigating the function f for a conditional extremum is reduced in this case to the problem of investigating Lagrange's function $F = f + \sum_{i=1}^{k} \lambda_i \varphi_i$ for an unconditional extremum.

The rule of Lagrange's factors is based on the necessary condition for an extremum.

THEOREM 14. *The point of conditional extremum of function f is a stationary point of Lagrange's function.*

In order to clarify the question, whether a conditional-stationary point is really a point of conditional extremum, we make use of the following theorem.

THEOREM 15. *If at a conditional-stationary point X^0 of a function f defined in the manifold E*

$$\varphi_i(x_1, x_2, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, k), \tag{2.65}$$

the second differential of Lagrange's function

$$J = \sum \frac{\partial^2 F}{\partial x_m \, \partial x_j} h_m h_j \tag{2.66}$$

is a form, which is which is defined to be positive (negative) in the linear $(n-k)$-dimensional manifold,

$$\sum_{m=1}^{n} \frac{\partial \varphi_i}{\partial x_m} h_m = 0 \quad (i = 1, 2, \ldots, k), \tag{2.67}$$

then X^0 is the point of conditional minimum (maximum).

§ 6. Stationary Points

1. Suppose G is a region in the n-dimensional space E_n, in which the function $f(X) = f(x_1, x_2, \ldots, x_n)$ of the class C_2 is defined, i.e. a function, which has continuous derivatives up to the second order inclusive. The point X^0 $(x_1^0, x_2^0, \ldots, x_n^0)$ of the region G is called *stationary* if at that point (see (2.53))

$$\frac{\partial f(x_1^0, x_2^0, \ldots, x_n^0)}{\partial x_i} = 0 \quad (i = 1, 2, \ldots, n). \tag{2.68}$$

The points of relative minimum and maximum of f in G are stationary. As will be seen below, there exist stationary points of another kind.

The property of the point X^0 to be stationary for a given function is preserved when the variables are changed: if

$$x_i = \varphi_i(y_1, y_2, \ldots, y_n) \quad (i = 1, 2, \ldots, n)$$

and, in particular, $x_i^0 = \varphi_i(y_1^0, y_2^0, \ldots, y_n^0)$, then the function $f(X)$ is represented in the form

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= f[\varphi_1(y_1, \ldots, y_n), \ldots, \varphi_n(y_1, \ldots, y_n)] \\ &= \bar{f}(y_1, \ldots, y_n). \end{aligned}$$

From the rule of differentiation of a complex function and from (2.68) it follows, that

$$\frac{\partial \bar{f}(y_1^0, y_2^0, \ldots, y_n^0)}{\partial y_i} = 0 \quad (i = 1, 2, \ldots, n). \tag{2.69}$$

Denote by z_0 the value of the function at a stationary point

$$z_0 = f(X^0) = f(x_1^0, x_2^0, \ldots, x_n^0). \tag{2.70}$$

Then, putting $f(X^0 + h) = f(x_1^0 + h_1, x_2^0 + h_2, \ldots, x^0 + h_n)$ and regarding $X^0 + h \in G$, we have

$$f(X^0 + h) = z_0 + \tfrac{1}{2} d^2 f(X^0, h) + o(\|h\|^2), \tag{2.71}$$

where

$$\frac{1}{2} d^2 f(X^0, h) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f(X^0)}{\partial x_i \, \partial x_j} h_i h_j. \tag{2.72}$$

A determinant of the nth order composed of second derivatives of the function f at the point X^0,

$$H(X^0) = \left| \frac{\partial^2 f(X^0)}{\partial x_i \, \partial x_j} \right| \quad (i, j = 1, 2, \ldots, n), \tag{2.73}$$

is called *Hesse's determinant* or the *Hessian* of the function $f(X)$ at the point X^0. The stationary point X^0 is called *non-degenerate* if $H(X^0) \neq 0$, and *degenerate* if the Hessian at it equals 0. We shall confine ourselves to the consideration of non-degenerate stationary points.

For the case of a function $f(x)$ of one variable, a point at which $f'(x) = 0$ is a stationary point. The condition of non-degeneracy has the form $f''(x) \neq 0$, since Hesse's determinant coincides here with the second derivative. Even in this simplest case, the investigation of a degenerate stationary point at which $f''(x) = 0$ presents additional difficulties.

The property of non-degeneracy of a point is preserved under a transformation non-degenerate in the neighbourhood of X^0, of variables of class C_2.

2. By means of the linear transformation

$$\eta_i = \sum_{j=1}^{n} a_j h_j \quad (i = 1, 2, \ldots, n)$$

it is possible to reduce the quadratic form (2.72) to a sum of squares

$$\frac{1}{2} \sum_{i,j} \frac{\partial^2 f(X)}{\partial x_i \, \partial x_j} h_i h_j = \sum_{i=1}^{n} \alpha_i \eta_i^2 . \tag{2.74}$$

From the condition of non-degeneracy $H(X^0) \neq 0$ it follows that all $a_i \neq 0$. If the coordinates of the vector h in the new system are $\eta_1, \eta_2, \ldots, \eta_n$, then it follows from (2.71) and (2.74) that

$$f(X^0 + h) = z_0 + \sum_{i=1}^{n} \alpha_i \eta_i^2 + o(\|h\|^2). \tag{2.75}$$

The point X^0 is called *a stationary point of the order k* $0 \leqslant k \leqslant n$, if among the coefficients α_i there are k negative ones, and so $n - k$ positive ones. The property of point X^0 in being stationary of the kth order is preserved on the non-degenerate transformation of variables. The stationary point of order 0 is a *minimal point*, and the stationary point of the order n is a *maximal point* of a given function.

In the case of a function of one variable, for $f'' > 0$ we obtain a point of zero order (minimum) and for $f'' < 0$ we get a point of the first order (maximum). In the number of arguments of the function, $n > 1$, there may also appear stationary points of an intermediate order k, $1 < k < n$, other than maximum or minimum.

Let X^0 be a stationary point of the intermediate order k. Then the n-dimensional space E_n can be represented in the form of a straight sum $E_k + E_{n-k}$ in such a way that for $h \in E_k$ the function $f(X^0 + h)$ has a minimum at X^0 (when $h = \theta$), and for $h' \in E_{n-k}$ the function $f(X^0 + h')$ has a maximum at the point X^0 (when $h' = \theta$). Such points are called *points of minimax of order k*.

When $n = 2$ the non-degenerate stationary points $X^0 = (x_0, y_0)$ of function $f(X) = f(x, y)$ are those in which

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0, \quad H(x^0, y^0) = \left[\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2\right]_{X^0} \neq 0. \tag{2.76}$$

They may be of the order $k = 0, 1, 2$.

A stationary point of the order $k = 0$ is a minimal point.

The conditions for that are

$$\frac{\partial^2 f}{\partial x^2} > 0, \quad H(x^0, y^0) > 0, \tag{2.77}$$

whence it follows also that $\partial^2 f/\partial y^2 > 0$. If $f(x^0, y^0) = z_0$, the point $X^0(x^0, y^0)$, for the level-line $f = z_0$, is *isolated*. In the neighbourhood of this point the lines $f = z$ are absent, when $z < z_0$, and the level-lines $f = x'$, when $z' > z_0$, approximate to ellipses with centre at X^0.

When $k = 1$ we have a minimax point. The condition for it is the inequality

$$H(X^0) < 0. \tag{2.78}$$

The point X^0 in this case is a *double point* of the level line $f = z_0$. This line divides the neighbourhood of the point X^0 into *four* parts, A_1, A_2, A_3, A_4, of which A_1 and A_3 are filled by level-lines $f = z$ when $z > z_0$, and A_2 and A_4 by level-lines $f = z'$ when $z' < z_0$.

Finally, when $k = 2$ we have a maximal point. The conditions for that are

$$\frac{\partial^2 f}{\partial x^2} < 0, \quad H(X^0) > 0, \tag{2.79}$$

and so, also, $\partial^2 f/\partial y^2 < 0$. If $f(x^0, y^0) = z_0$, then the level-lines $f = z'$ are absent when $z' > z_0$, and the level-lines $f(z)$ when $z < z_0$ approximate to ellipses with a centre at the point X^0.

3. A topological investigation of the behaviour of a function in the neighbourhood of stationary points of any order was carried out by M. Morse. He also proved the theorem about the ratio of the numbers of stationary points of different orders. Let us give the simplest case of this theorem.

Suppose G is a bounded plane region, whose boundary q is a smooth closed curve, and $f(x, y)$ is a function of class C_2 having only non-degenerate stationary points inside G and having no stationary points on q. Suppose, further, that $f(x, y) = \text{const}$ on q, i.e. q is a level-line for f (or its part). Denote the number of stationary points of the order i of the function f in G by m_i ($i = 0, 1, 2$). Then the following theorem holds.

THEOREM 16. *The numbers m_0, m_1, m_2 are connected by means of the relationship*

$$m_0 - m_1 + m_2 = 1,$$

i.e. the number of extremal points of a function of two variables is greater by one than the number of its minimax points.

A visual illustration of this theorem is afforded by the following. Suppose G is an island and the function f denotes the height of points on the island above sea level. Then m_2 is the number of all peaks, m_0 is the number of the deepest points and m_1 is the number of cols (or saddle points). The theorem quoted above states that the number of cols is less by one than the combined number of peaks and depressions on the island.

CHAPTER III

COMPOSITE AND IMPLICIT FUNCTIONS OF n VARIABLES

§ 1. Transformation of Variables. Composite Functions

1. Let there be defined in the region G of space E_n the continuous operator

$$Y = f(X), \tag{3.1}$$

which maps the region G into space of the same number of dimensions. Being given such an operator is equivalent to being given a set of n continuous functions of n variables:

$$y_i = f_i(x_1, x_2, \dots, x_n) \quad (i = 1, 2, \dots, n). \tag{3.2}$$

The set D of all possible values of Y is a region, which is called the *image* of the region G for the given mapping. In its turn, the region G is called the *inverse image* or *original* of the region D.

The operator f from E_n into E_n can be regarded as a transformation of coordinates in space E_n, i.e. as a transition from the system of coordinates $(x_1, x_2, \dots, x_n)$ to the system $(y_1, y_2, \dots, y_n)$. It is usually assumed that the image D together with its original G, or at least one of these regions, coincides with the whole of the space E_n.

The simplest examples of such transformations are:

(a) The transition from *polar* coordinates to cartesian ones, determined by the pair of functions

$$\left.\begin{aligned} y_1 &= x_1 \cos x_2, \\ y_2 &= x_1 \sin x_2 \end{aligned}\right\} \tag{3.3}$$

in the plane E_2.

(b) The transition, in space E_3, from *cylindrical* coordinates to cartesian ones, determined by the system of functions

$$\left.\begin{aligned} y_1 &= x_1 \cos x_2, \\ y_2 &= x_1 \sin x_2, \\ y_3 &= x_3. \end{aligned}\right\} \tag{3.4}$$

(c) The transition, in space E_3, from *spherical* coordinates to cartesian ones, determined by the system of functions

$$\left.\begin{aligned} y_1 &= x_1 \cos x_2 \sin x_3, \\ y_2 &= x_1 \sin x_2 \sin x_3, \\ y_3 &= x_1 \cos x_3. \end{aligned}\right\} \tag{3.5}$$

These examples are considered in Chapter IV. Other examples are also considered there.

2. In a more general case the system

$$y_i = f_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, m) \tag{3.6}$$

of m differentiable functions of n variables establishes a correspondence between the vector $X(x_1, x_2, \ldots, x_n)$ of space E_n and the vector $Y(y_1, y_2, \ldots, y_m)$ of space E_m, i.e. it determines the continuous operator mapping E_n in E_m.

If $m < n$ the mapping is done into space of fewer dimensions. In this case the original of every point $D \in E_m$ is, in general, a set in space E_n, which has dimensions $n - m$. In particular, when $m = 1$ one function of n variables maps E_n in a straight line. The inverse image of a point of a straight line is an equipotential hypersurface. In mapping into space of a smaller number of dimensions, it is possible that the image D of the region $G \subset E_n$ be made to coincide with the whole space E_m.

In mapping E_n into E_m, when $m > n$ the image D of the region $G \subset E_n$ cannot coincide with the whole space E_m even when $G = E_n$. The region D is necessarily a manifold of n dimensions in E_m. So, when $n = 1$, the continuous operator maps a segment $G \subset E_1$ into a *line* of space E_m, for example, into a plane curve, when $m = 2$. When $n = 2$ and $m = 3$, the continuous image of region G of a plane is a *surface, given parametrically.*

3. If a function $z = \varphi(Y)$ is given in the region D, the operator $Y = f(X)$, which maps the region G into its image D, maps this function into the function

$$z = \varphi[f(X)] = \psi(X), \tag{3.7}$$

defined in the region G. Such a function is called *composite*. In order that the composite function $\psi(X) = \varphi[f(X)]$ be differentiable, it is sufficient that the function $\varphi(Y)$ and the operator $f(X)$ be differentiable.

The operator $Z = \varphi(Y)$ defined in the region D is transformed by the operator $Y = f(X)$ into a new operator

$$Z = \Phi[f(X)] = \Psi(X), \tag{3.8}$$

which is defined in the region G. The composite function introduced above can be regarded as a particular case of such a compound operator. Here, if the operator $f(X)$ maps E_n into E_m and operator $\Phi(Y)$ maps E_m into E_k, then the compound operator $\psi(X)$ maps space E_n into E_k. A composite function is obtained when $k = 1$.

The Jacobi matrix of a compound operator $\psi(X)$ can be obtained from the Jacobi matrices of operators $f(X)$ and $\Phi(Y)$ (see p. 64) with the help of the rule of multiplying matrices, according to the formula

$$(\Psi) = (\Phi)(f). \tag{3.9}$$

For example, when $n = m = k = 2$, we have

$$\begin{Vmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{Vmatrix} = \begin{Vmatrix} \dfrac{\partial z_1}{\partial y_1} & \dfrac{\partial z_1}{\partial y_2} \\ \dfrac{\partial z_2}{\partial y_1} & \dfrac{\partial z_2}{\partial y_2} \end{Vmatrix} \cdot \begin{Vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{Vmatrix}. \tag{3.10}$$

When $k = 1$ the compound operator is a composite function and its Jacobi matrix degenerates into a row-matrix consisting of partial derivatives. Formula (3.9) gives

$$\begin{Vmatrix} \dfrac{\partial z}{\partial x_1} \dfrac{\partial z}{\partial x_2} \cdots \dfrac{\partial z}{\partial x_n} \end{Vmatrix} = \begin{Vmatrix} \dfrac{\partial z}{\partial y_1} \dfrac{\partial z}{\partial y_2} \cdots \dfrac{\partial z}{\partial y_n} \end{Vmatrix} \begin{Vmatrix} \dfrac{\partial y_1}{\partial x_1} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial y_1}{\partial x_n} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{Vmatrix}. \tag{3.11}$$

Hence follows a formula for the partial derivatives of a composite function

$$\frac{\partial z}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_i}. \tag{3.12}$$

If the vector dX is the increment X, then the differential of the operator $\psi(X)$, as was shown in § 4 of the preceding chapter, has the form

$$d\Psi(X, dX) = \Psi'(X)\, dX, \tag{3.13}$$

where $\Psi'(X)$ denotes the Jacobi matrix (Ψ) of the operator $\Psi(X)$. Using an analogous equation

$$dY = df(X, dX) = f'(X)\, dX$$

and formula (2.9) we bring the expression for the differential of a compound operator to the form

$$dZ = \Phi'(Y)\, dY, \tag{3.14}$$

where dY denotes not the increment of the operator Y, but its differential.

Equation (3.14) shows that the differential of the operator preserves its form also in the case when the operator is a compound one, i.e. when its argument in its turn is an operator of some other independent vector. This property is called *the invariancy of the differential.* The invariancy of the differential holds, in particular, also for a composite function (i.e. when $k = 1$).

4. Suppose $Y = f(X)$ is an operator mapping E_n into E_n. Then the Jacobi matrix of this operator is quadratic. The determinant of this matrix is called *Jacobi's determinant* or *Jacobian* of the image. The Jacobian of the system of functions (3.2) is usually denoted by the symbol

$$\frac{D(y_1, y_2, \ldots, y_n)}{D(x_1, x_2, \ldots, x_n)}, \tag{3.15}$$

similar to the notation of a derivative.

Let us consider the operator $Z = \Phi(Y)$ also mapping E_n into E_n. Since the determinant of the product of the matrices equals the product of the determinants of the terms to be multiplied, we get from (3.9) for the Jacobian of a compound operator

$$\frac{D(z_1, z_2, \ldots, z_n)}{D(x_1, x_2, \ldots, x_n)} = \frac{D(z_1, z_2, \ldots, z_n)}{D(y_1, y_2, \ldots, y_n)} \frac{D(y_1, y_2, \ldots, y_n)}{D(x_1, x_2, \ldots, x_n)}. \tag{3.16}$$

A more general formula: let the operator $Y = f(X)$ map E_n into E_m and the operator $Z = \Phi(Y)$ map E_m into E_n, and $m > n$. Then,

$$\frac{D(z_1, z_2, \ldots, z_n)}{D(x_1, x_2, \ldots, x_n)}$$

$$= \sum_{i_1, i_2, \ldots, i_n} \frac{D(z_1, z_2, \ldots, z_n)}{D(y_{i_1}, y_{i_2}, \ldots, y_{i_n})} \frac{D(y_{i_1}, y_{i_2}, \ldots, y_{i_n})}{D(x_1, x_2, \ldots, x_n)}, \quad (3.17)$$

where the sum extends over all possible groups of the indices 1, 2, ..., m, n in each group. In particular, when $n = 2$, $m = 3$ we get

$$\frac{D(z_1, z_2)}{D(x_1, x_2)} = \frac{D(z_1, z_2)}{D(y_1, y_2)} \frac{D(y_1, y_2)}{D(x_1, x_2)} + \frac{D(z_1, z_2)}{D(y_2, y_3)} \frac{D(y_2, y_3)}{D(x_1, x_2)}$$

$$+ \frac{D(z_1, z_2)}{D(y_3, y_1)} \frac{D(y_3, y_1)}{D(x_1, x_2)}. \quad (3.18)$$

The analogy between the Jacobian and the derivative of a function of one variable is revealed by interpreting the Jacobian geometrically. Let G be some neighbourhood of the point X, and D be the image of G on mapping by $Y = f(X)$. Denote the n-dimensional volumes of regions G and D by $V_n(G)$ and $V_n(D)$ respectively. Then the Jacobian of the mapping at point X equals *the limit of the ratio of volumes* $V_n(D)/V_n(G)$, when the neighbourhood G contract to a point. In particular, when $n = 2$ the Jacobian is the coefficient of distortion of areas in mapping.

5. The variables $z_1, z_2, \ldots, z_n$ can turn out to be identical to the variables $x_1, x_2, \ldots, x_n$. In this case, we are speaking of the *inverse* of the operator $Y = f(X)$. Care should be taken to investigate whether this inverse does exist.

THEOREM 1. *If the Jacobian of the mapping $Y = f(X)$ differs from zero at some point $X(x_1, x_2, \ldots, x_n)$, is continuous at that point and in its neighbourhood, and*

$$\frac{D(y_1, y_2, \ldots, y_n)}{D(x_1, x_2, \ldots, x_n)} \neq 0,$$

then there exists a neighbourhood G of this point, in which the mapping is a one-to-one correspondence and the inverse exists.

This means that every point of the image D of the region G has as its original in G one point and one only. The Jacobian of the inverse mapping can be found from the formula

$$\frac{D(x_1, x_2, \ldots, x_n)}{D(y_1, y_2, \ldots, y_n)} = \frac{1}{\dfrac{D(y_1, y_2, \ldots, y_n)}{D(x_1, x_2, \ldots, x_n)}}. \tag{3.19}$$

The theorem about the inversion of mapping is a particular case of the theorem about the existence of an implicit function.

§ 2. Implicit Functions. Functions Dependent on a Parameter

1. Suppose that the values of variables x and y are connected by an equation of the form

$$F(x, y) = 0.$$

If to every value of x in some interval there correspond one or several values of y, satisfying this equation together with x, then this equation defines the (single-valued or multiple-valued) function $y = f(x)$ turning the equation $F(x, y) = 0$ into an identity. For example, the equation of a circle $x^2 + y^2 = a^2$ defines a two-valued function $y = \pm\sqrt{a^2 - x^2}$. It is said that the equation $F(x, y) = 0$ defines *an implicit function* of one variable.

The conditions of existence, continuity and differentiability of an implicit function of one variable are established by means of the following theorem.

THEOREM 2. *Suppose $F(x, y)$ is a function of two variables, continuously differentiable in some neighbourhood of point (a, b). If $F(a, b) = 0$ and $F'(a, b) \neq 0$ then there exists a number $\delta > 0$ such that the equation $F(x, y) = 0$ defines, in the interval $(a - \delta, a + \delta)$, a single-value, continuous and differentiable function $y = f(x)$, transforming this equation into an identity and satisfying the equation $b = f(a)$.*

The derivative of an implicit function can be found in accordance with the formula

$$y' = -\frac{F'_x(x, y)}{F'_y(x, y)}. \tag{3.20}$$

The following derivatives can be found by using formulae:

$$\left.\begin{aligned} y'' &= -\frac{F''_{xx}(F'_y)^2 - 2F''_{xy}F'_xF'_y + F''_{yy}(F'_x)^2}{(F'_y)^3}, \\ y''' &= -(F'_y)^{-5}[F'''_{xxx}(F'_y)^4 - 3F'''_{xxy}(F'_y)^3F'_x \\ &\quad + 3F'''_{xyy}(F'_y)^2(F'_x)^2 - F'''_{yyy}F'_y(F_x)^3 - 3F''_{xx}F''_{xy}(F'_y)^3 \\ &\quad + 3F''_{xx}F''_{yy}F'_x(F'_y)^2 + 6(F''_{xy})^2F'_x(F'_y)^2 \\ &\quad + 3(F''_{yy})^2(F'_x)^3 - 9F''_{xy}F''_{yy}(F'_x)^2F'_y], \quad \text{etc.} \end{aligned}\right\} \tag{3.21}$$

They are all obtained by means of the differentiation of the identity $F(x, y) = 0$ as a composite function and by utilizing derivatives known already. It is sufficient for the existence of any of them, that the right-hand side should have a meaning, i.e. that all its constituent derivatives should exist and the denominator should not equal zero.

In geometric terms, the line of zero level of the function $F(x, y) = 0$ should be represented in the form of a graph of the explicitly given function $y = f(x)$. This is, it seems, possible locally, i.e. in the neighbourhood of any point except those in which the tangent to the level-line is parallel to the y-axis.

2. An analogous situation exists also in the case of one implicit function of several variables. The equation

$$F(x_1, x_2, \dots, x_n, y) = 0 \tag{3.22}$$

defines an n-dimensional equipotential hypersurface of the function of n variables in $(n+1)$-dimensional space. If to every vector $X(x_1, x_2, \dots, x_n)$ there corresponds a value y, satisfying, together with X, the equation (3.22), the latter defines an implicit function of vector X—an implicit function of n variables.

The conditions of existence of such an implicit function do not differ in substance from the conditions of existence of an implicit function of one variable.

THEOREM 3. *Suppose $F(x_1, x_2, \dots, x_n, y)$ is a function of $(n+1)$ variables continuously differentiable in some neighbourhood of the point $X(x_1^0, x_2^0, \dots, x_n^0, y^0)$. If $F(x_1^0, x_2^0, \dots, x_n^0, y^0) = 0$ and $F'(x_1, x_2, \dots, x_n, y) \neq 0$ at the point under consideration, then there exists a number $\delta > 0$ such that the equation $F(x_1, x_2, \dots, x_n, y) = 0$ defines, in the δ-neighbourhood of the point X, a single-value continuous and differentiable function $y = f(X)$, which transforms this*

equation into an identity and satisfies the equation $y^0 = f(x_1^0, x_2^0, \ldots, x_n^0)$.

Partial derivatives of this function are found according to the same formulae (3.20), (3.21).

3. In the most general case it is possible to consider a system of m equations with $n + m$ variables

$$\left.\begin{aligned} F_1(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) &= 0, \\ F_2(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_m(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) &= 0 \end{aligned}\right\} \tag{3.23}$$

as the operator

$$F(X, Y) = 0, \tag{3.24}$$

which maps the $(n + m)$-dimensional vector or, which is the same, a pair of vectors of dimensionality n and m respectively, into a zero vector of m-dimensional space.

The system (3.23) defines $y_1, y_2, \ldots, y_m$ as implicit functions of variables $x_1, x_2, \ldots, x_n$. In other words, the operator (3.24) defines the m-dimensional vector Y as an implicit function of the n-dimensional vector X. Conditions of existence of an implicit function are analogous to the conditions of Theorems 2 and 3, while the role of the derivative is taken here by the Jacobian

$$\frac{D(F_1, F_2, \ldots, F_m)}{D(y_1, y_2, \ldots, y_m)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial y_1} & \ldots & \dfrac{\partial F_1}{\partial y_m} \\ \ldots & \ldots & \ldots \\ \dfrac{\partial F_m}{\partial y_1} & \ldots & \dfrac{\partial F_m}{\partial y_m} \end{vmatrix}. \tag{3.25}$$

THEOREM 4. *Suppose, the functions* $F_1, F_2, \ldots, F_m$ of $n + m$ *variables* $x_1, x_2, \ldots, x_n, y_1, \ldots, y_m$ *are continuously differentiable in the neighbourhood of point* $x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0$ *and have the Jacobian* $\dfrac{D(F_1, F_2, \ldots, F_m)}{D(y_1, y_2, \ldots, y_m)}$ *non-zero at that point. Then there exists a neighbourhood of the point* $(x_1^0, x_2^0, \ldots, x_n^0)$ (*vector* X^0) *in which the system of equations* (3.23) (*operatorial equation* (3.24)) *defines m functions*

$$y_i = f_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, m) \tag{3.26}$$

(operator $Y = f(X)$), continuous and differentiable in the neighbourhood of this point and such that

$$y_i^0 = f_i(x_1^0, x_2^0, \ldots, y_n^0) \quad (i = 1, 2, \ldots, m)$$

$$(Y^0 = f(X^0)).$$

The elements of the Jacobi matrix of the operator $Y = f(X)$, definable implicitly by means of the equation (3.24), are found according to formulae

$$\frac{\partial y_i}{\partial x_j} = -\frac{\dfrac{D(F_1, F_2, \ldots, F_m)}{D(y_1, \ldots, y_{i-1}, x_j, y_{i+1}, \ldots, y_m)}}{\dfrac{D(F_1, F_2, \ldots, F_m)}{D(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_m)}} \tag{3.27}$$

$$(i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n).$$

Equation (3.27) demonstrates once again the analogy between the Jacobian and the derivative of a function of one variable.

Considering in particular n equations with $2n$ unknowns

$$F_i(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \quad (i = 1, 2, \ldots, n), \tag{3.28}$$

we find

$$\frac{D(y_1, y_2, \ldots, y_n)}{D(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\dfrac{D(F_1, F_2, \ldots, F_n)}{D(x_1, x_2, \ldots, x_n)}}{\dfrac{D(F_1, F_2, \ldots, F_n)}{D(y_1, y_2, \ldots, y_n)}}. \tag{3.29}$$

If we assume equations (3.28) solved with respect to variables $y_1, \ldots, y_n$, the theorem of existence of an implicit function becomes a theorem about the possibility of inversion of the operator given in § 1, sec. 5. Formula (3.29) contains the formula (3.19) quoted there as a particular case.

4. Jacobi's matrix can be used also in the discussion of the dependence of functions. Suppose

$$y_i = F_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, m) \tag{3.30}$$

is a set of m functions of n variables. The functions together with their partial derivatives are assumed to be continuous in some region G.

The function y_i is called *dependent on the remaining functions of the set* if in G the relationship

$$y_j = \varphi(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m). \tag{3.31}$$

is fulfilled identically with respect to $x_1, x_2, \ldots, x_n$.

The functions $y_1, y_2, \ldots, y_m$ are called *dependent* in the region G if any one of them depends on the others.

If, on the contrary, neither in the region G nor in any of its parts does there occur an identity of form (3.31), it is said that functions $y_1, \ldots, y_m$ are *independent* in G.

In order to establish the independence of the system of functions (3.30) it is desirable to turn to Jacobi's matrix of this system. The following theorem holds:

THEOREM 5. *In order that the system of functions* (3.30), *when* $m < n$, *be independent, it is necessary, and sufficient, that at least one of the determinants of the m-th order, constructed out of the element of the Jacobi matrix of the system, be non-zero in the region G.*

A more exact result can be obtained using the concept of the rank of Jacobi's matrix. We give the name *rank* to the highest of the orders of the determinants formed from the Jacobi matrix and not becoming zero identically in G. In other words, if the rank of a matrix equals μ, then $\mu \leqslant n$, $\mu \leqslant m$ and there exists in the matrix a determinant of the order μ, differing from an identical zero, while all the determinants of orders higher than μ, which can be isolated from the given Jacobi matrix, equal zero identically.

THEOREM 6. *A system of m functions (*3.30*) contains exactly* μ *independent ones* ($\mu \leqslant m$, $\mu \leqslant n$) *in the region G or in some part of it, if, and only if, the Jacobi matrix of the system has the rank* μ.

5. A particular case of implicit functions are functions definable by means of equations dependent on a parameter.

Suppose, for example, X is a vector of n-dimensional space and $F(X)$ is an operator mapping E_n into E_n. The equation

$$F(X) = 0 \tag{3.32}$$

defines a fixed vector in E_n. If the operator depends on the parameter, the equation

$$F_t(X) = 0 \tag{3.33}$$

defines the vector X, which is a function of the parameter t, $X = X(t)$.

THEOREM 7. *If the continuous operator F depends continuously on the parameter t and the Jacobian of the operator is non-zero for some value $t = t_0$ and in its neighbourhood, the solution $X = X(t)$ of the equation* (3.33) *is a continuous function of the parameter t in some neighbourhood of t_0.*

The situation does not alter if there are several parameters.

6. The theorems of the preceding points, relating only to first derivatives, can be extended to cover a more general case. Namely, let functions $F_1, F_2, \ldots, F_m$ of $n + m$ variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$ belong to the class C_k in the neighbourhood of the point $x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_m^0$ and have a Jacobian $D(F_1, \ldots, F_n)/D(y_1, \ldots, y_m)$, non-zero at that point. Then the set of equations (2.23)

$$F_1(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_m(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$$

defines in some neighbourhood of point $(x_1^0, \ldots, x_n^0)$ the system of m functions

$$y_i = f(x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, m), \tag{3.34}$$

also belonging to the class C_k. This proposition can be written down in operational form.

THEOREM 8. *If the operator F belongs to the class C_k, the equation (3.24)*

$$F(X, Y) = 0$$

defines the m-dimensional vector Y as an implicit function belonging to the class C_k of the n-dimensional vector X in some neighbourhood of vector X^0, on condition that the Jacobian $D(F)/D(Y)$ is non-zero for $X = X^0$, $Y = Y^0$.

By analogy with the above, if the functions F_i $(i = 1, 2, \ldots, m)$ are *analytic*, i.e. capable of being represented by power series (the operator F is analytic), the system of equations (3.23) (operatorial equation (3.24)), when the same assumptions are made with respect to the Jacobian, defines a system of implicit functions (operator) which are *analytic*. The latter proposition remains true in the case when we are dealing with functions of a complex variable.

The Method of Indeterminate Coefficients

7. Suppose we are given the equation

$$F(x, y) = 0. \tag{3.35}$$

We shall regard x as a parameter and y as a root of the equation (3.35) which depends on the parameter x. Let F be an analytical function of variables x and y (which can be regarded as complex for the time being) in the neighbourhood of the point (0, 0). Further, we put down $F(0, 0) = 0$ and

$$F'_y(0, 0) \neq 0. \tag{3.36}$$

On the basis of the theorem about implicit functions in the neighbourhood of the point (0, 0), the equation (3.35) defines a single-value analytical function $y = y(x)$, capable of being represented, for sufficiently small $|x|$ by the power series

$$y(x) = \sum_{k=1}^{\infty} a_k x^k. \tag{3.37}$$

Take the expansion

$$F(x, y) = \sum_{r,\, s=0}^{\infty} b_{rs} x^r y^s. \tag{3.38}$$

From (3.36) it follows that

$$b_{00} = 0, \quad b_{01} \neq 0. \tag{3.39}$$

Substituting (3.37) into (3.38), we get, on the basis of (3.35),

$$0 = F(x, y) = \sum_{r,\, s=0}^{\infty} b_{r,s} x^r \left(\sum_{k=1}^{\infty} a_k x^k \right)^s \equiv \sum_{n=1}^{\infty} c_n x^n, \tag{3.40}$$

where the coefficients c_n ($n = 1, 2, \ldots$) are expressed in terms of $b_{r,s}$ and a_k according to the formulae

$$c_1 = b_{10} + b_{01} a_1, \tag{3.41$_1$}$$

$$c_2 = b_{20} + b_{01} a_2 + b_{02} a_1^2 = b_{01} a_2 + z_2, \tag{3.41$_2$}$$

. . . .

$$c_n = b_{01} a_n + z_n \tag{3.41$_n$}$$

(where $z_2 = b_{20} + b_{02} a_1^2$ depends on b_{20}, b_{02} and a_1, and z_n depends on certain coefficients b_{rs}, and also on $a_1, a_2, \ldots, a_{n-1}$).

It follows from (3.40) that all coefficients c_n equal zero. Then from (3.41) we get

$$a_1 = -\frac{b_{10}}{b_{01}}$$

which is possible as a corollary of (3.39). Having found a_1, we determine z_2, and from the condition $c_2 = 0$ on the basis of (3.41_2) and (3.39) we find a_2.

We obtain a recurrent process of consecutive definition of coefficients $a_1, a_2, \ldots, a_n, \ldots$, of the expansion (3.37). Suppose $a_1, a_2, \ldots, a_{n-1}$ are known already, then, from (3.41) it is possible to determine z_n, and then, taking into account the equation $c_n = 0$, to find a_n.

The series (3.37) with the coefficients thus found is convergent for sufficiently small $|x|$ and represents the function y.

If all coefficients b_{rs} are real numbers, then the numbers a_k are also real and the expansion (3.37) is the expansion of the function $y(x)$ in the real domain.

If the function $F(x, y)$ belongs to the class C_n, i.e. is capable of being represented in the form

$$F(x, y) = \sum_{r,\, s=0}^{n} b_{rs}x^r y^s + r_n, \tag{3.42}$$

where $r_n = o(|x| + |y|)^n$, $b_{00} = 0$, $b_{01} \neq 0$, then from the equation $F(x, y) = 0$, it is possible to find, in the same way, the coefficients $a_1, a_2, \ldots, a_n$ of the representation

$$y = y(x) = \sum_{k=1}^{n} a_k x^k + \varepsilon, \quad \varepsilon = o(|x|^n).$$

EXAMPLE 1. The equation

$$F(x, y) \equiv x + y + xy + x^3y^2 + y^5 = 0$$

is given.

Here $F(0, 0) = 0$, $F'(0, 0) = 1 \neq 0$. It follows, from the theorem about implicit functions in the neighbourhood of the point (0, 0), that for sufficiently small $|x|$, y can be represented by the series

$$y = \sum_{k=1}^{\infty} a_k x^k.$$

On substituting this expression in the original equation and reducing similar terms, we get

$$0 = F(x, y) = \sum_{n=1}^{\infty} c_n x^n \equiv (a_1 + 1)x + (a_2 + a_1)x^2 + (a_3 + a_2)x^3 + (a_4 + a_3)x^4$$
$$+ (a_5 + a_4 + a_1^2 + a_1^5)x^5 + (a_6 + a_5 + 2a_1a_2 + 5a_1^4a_2)x^6$$
$$+ (a_7 + a_6 + a_2^2 + 2a_1a_3 + 5a_1^4a_3 + 10a_1^3a_2^2)x^7$$
$$+ (a_8 + a_7 + 2a_1a_4 + 2a_2a_3 + a_1^4a_4 + 16a_1^3a_2a_3 + 10a_1^2a_3^3)x^8 + \cdots,$$

whence we find the set equations for the determination of $a_1, a_2, \ldots$:

$$\begin{aligned}
c_1 &\equiv a_1 + 1 = 0,\\
c_2 &\equiv a_2 + a_1 = 0,\\
c_3 &\equiv a_3 + a_2 = 0,\\
c_4 &\equiv a_4 + a_3 = 0,\\
c_5 &\equiv a_5 + a_4 + a_1^2 + a_1^5 = 0,\\
c_6 &\equiv a_6 + a_5 + 2a_1a_2 + 5a_1^4a_2 = 0,\\
c_7 &\equiv a_7 + a_6 + a_2^2 + 2a_1a_3 + 5a_1^4a_3 + 10a_1^3a_2^2 = 0,\\
c_8 &\equiv a_8 + a_7 + 2a_1a_4 + 2a_2a_3 + a_1^4a_4 + 16a_1^3a_2a_3 + 10a_1^2a_3^3 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Solving this system consecutively, we get

$$a_1 = -1, \quad a_2 = 1, \quad a_3 = -1, \quad a_4 = 1, \quad a_5 = -1, \quad a_6 = -2,$$
$$a_7 = 14, \quad a_8 = -17, \quad \ldots$$

and so, in the neighbourhood of the point (0, 0) we have

$$y = -x + x^2 - x^3 + x^4 - x^5 - 2x^6 + 14x^7 - 17x^8 + \cdots$$

§ 3. Newton's Diagram

1. The method of indeterminate coefficients described above does not apply if the condition (3.36) is infringed, i.e. if $F'(0, 0) = b_{01} = 0$. Indeed, in the equation (3.41) the coefficient of a_1 equals 0, and it is insoluble in the general case. In this case, for a given x, the equation (3.35) has, in general, several roots $y = y(x)$, which can be expanded into series of powers of $x^{1/q}$, and the integer q is different for different roots. For example, when $F(x, y) = y^5 - y^3x - y^2x + x^2$, the equation $F(x, y) = 0$ has five roots $y = y(x)$:

$$y_1 = x^{1/2} \quad y_2 = -x^{1/2}, \quad y_3 = x^{1/3}, \quad y_4 = \omega x^{1/4},$$
$$y_5 = \omega^2 x^{1/3} \quad \left(\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

Of these, two can be expanded in powers of $x^{1/2}$ and three in powers of $x^{1/3}$.

The method of determining the denominators q in the expansions of roots $y = y(x)$ in powers of $x^{1/q}$ and of the coefficients of these expansions goes back to Newton and is named after him.

We shall confine ourselves to the case when the function $F(x, y)$ in (3.35) is a polynomial of degree N with respect to y:

$$F(x, y) = \sum_{s=0}^{N} F_s(x)\, y^s, \tag{3.43}$$

and

$$F_n(x) \not\equiv 0. \tag{3.44}$$

Suppose that each function $F_s(x)$ is capable of being represented by a series of powers of $x^{1/p}$, more generally

$$F_s(x) = x^{\varrho_s} \sum_{r=0}^{\infty} F_{rs} x^{r/p}. \tag{3.45}$$

Here, if $F_s(x) \not\equiv 0$, the first coefficient F_{0s} can be regarded as non-zero. Further, equations $\Phi(x, y) = 0$ and $y^{m\Phi}(x, y) = 0$ have equal roots $y = y(x)$ for every x, if we ignore the roots $y \equiv 0$. In so far as such roots are of no interest, it can be taken that

$$F_0(x) \not\equiv 0. \tag{3.46}$$

It follows from (3.44) and (3.46) that

$$F_{00} \neq 0, \quad F_{N0} \neq 0. \tag{3.47}$$

We shall seek solutions $y = y(x)$ of the equation (3.35) of the order x^ε, i.e. of the form

$$y = y_\varepsilon x^\varepsilon + Y, \tag{3.48}$$

where $y_\varepsilon \neq 0$ and $Y = 0(x^\varepsilon)$ when $x \to 0$. In order to determine possible values ε and y_ε it is necessary to substitute (3.48) in (3.55) and put the principal term, i.e. the coefficient of the lower power of x, equal to zero. While the index ε remains unknown, it is impossible to say which of these terms (after this substitution) are lower ones. It is clear, however, that terms of the lowest order are contained among the following ones:

$$F_{00} x^{\varrho_0},\ F_{0k} y_\varepsilon^k x^{\varrho_k + k\varepsilon},\ F_{0N} y_\varepsilon^N x^{\varrho_N + N\varepsilon}, \tag{3.49}$$

where k runs through those of the values $1, 2, \ldots, N-1$, for which $F_k(x) \not\equiv 0$. Since by (3.47) and (3.48) at least two coefficients $F_{0s} \neq 0$, therefore for the reduction of terms of the lowest order it is necessary to pick ε in such a way that at least two of the indices

$$\varrho_0, \quad \varrho_k + k\varepsilon, \quad \varrho_N + N\varepsilon$$

should coincide, and the remaining ones should not be smaller than these. This reasoning enables us to find all possible values of ε and their corresponding values of y_ε.

To find the values of ε Newton's diagram is used. Let us mark points with coordinates

$$(0, \varrho_0), \ (k, \varrho_k), \ (N, \varrho_N)$$

on a rectangular coordinate net, k going through the same values as in (3.49). We place a ruler alongside point $(0, \varrho_0)$ in such a way that it coincides with the axis of ordinates and we begin to rotate the ruler about point $(0, \varrho_0)$ anticlockwise, until it just reaches another of the points marked in, say (l, ϱ_l). The tangent of the angle made by the ruler and the negative direction of the axis of abscissae equals one of the possible values ε, because $\tan\alpha = (\varrho_0 - \varrho_l)/l = \varepsilon$. If we draw straight lines through points (s, ϱ_s) other than those which lie along the ruler, at the above angle, then these straight lines lie above the ruler and therefore $\varrho_s + s_\varepsilon > \varrho_l + l\varepsilon$.

Note that other points (k, ϱ_k) might lie alongside the ruler joining the points $(0, \varrho_0)$ and (l, ϱ_l).

We now rotate the ruler in the same direction about that point (l, ϱ_l) on the ruler, whose abscissa is greatest, until it coincides with some other one of the points marked in (p, ϱ_p). The tangent of angle between the new direction of the ruler and the negative direction of the axis gives another possible value $\tan\alpha = (\varrho_l - \varrho_p)/(p-l) = \varepsilon$ and straight lines passing through other points (s, ϱ_s) parallel to the given direction of the ruler lie higher, that means $\varrho_s + \varepsilon s > \varrho_l + \varepsilon l = \varrho_p + \varepsilon p$. Continuing this process we obtain all possible values of ε. The convex open polygon joining the points of rotation of the ruler is called *Newton's diagram.*

We now proceed to find the values of the coefficient of y_ε. Let (i, ϱ_i) and (j, ϱ_j) be the end-points of the segment of the diagram, which determines one of the possible values of ε. In order that, on

substitution of (3.48) into (3.35), the lower terms be eliminated it is necessary and sufficient that

$$P(y_\varepsilon) = \sum_{\substack{s \\ \varrho_s + s\varepsilon = \varrho_i + i\varepsilon}}' F_{0s} y_\varepsilon^s, \tag{3.50}$$

where the sign ′ beside the sum sign means that the summation is over only those values of s which satisfy the relationship shown

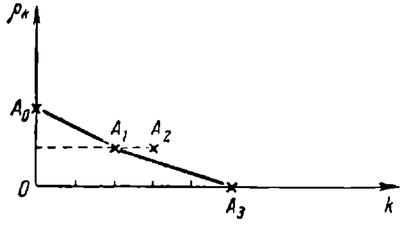

FIG. 8

under the sign of the sum. The equation (3.50) has $j - i$ non-zero roots (taking into account their multiplicity), i.e. as many roots as the length of the projection of the segment of the diagram taken. Hence it is seen that all N values of the principal term $y_\varepsilon x^\varepsilon$ in the expansion (3.46) can be obtained by this method.

In order to find the next term of the expansion of y it is necessary to substitute (3.48) in (3.35) and determine the lower term of the expansion by the same method, putting

$$Y = y_{\varepsilon'} x^{\varepsilon'} + o(x^{\varepsilon'}).$$

Continuing this process, we arrive at the following proposition (see ref. [33]).

THEOREM 9. *All N solutions of the equation* (3.35) *have the form*

$$y = y_\varepsilon x^\varepsilon + y_{\varepsilon'} x^{\varepsilon'} + y_{\varepsilon''} x^{\varepsilon''} + \cdots, \tag{3.51}$$

where $\varepsilon < \varepsilon' < \varepsilon'' \ldots$.

The numbers $\varepsilon, \varepsilon', \varepsilon'', \ldots$ are fractions with a finite common denominator. The series (3.51) are convergent in some neighbourhood of the point $x = 0$, except at the point $x = 0$ itself, if $\varepsilon < 0$.

To illustrate this method we consider two examples.

EXAMPLE 2. (See p. 90.)

$$F(x,y) \equiv x^2 - xy^2 - xy^3 + y^5 = F_0(x) + F_2(x)y^2 + F_3(x)y^3 + F_{\upsilon}(x)y^5,$$

$$F_0(x) = x^2 \quad (\varrho_0 = 2), \qquad F_2(x) = -x \quad (\varrho_2 = 1),$$

$$F_3(x) = -x \quad (\varrho_3 = 1), \qquad F_5(x) = 1 \quad (\varrho_5 = 0).$$

Construct points $A_0(0,2)$, $A_1(2,1)$, $A_2(3,1)$, $A_3(5,0)$ (Fig. 8). It can be seen from Newton's diagram that there are two values ε: $\varepsilon = 1/2$ (corresponding to

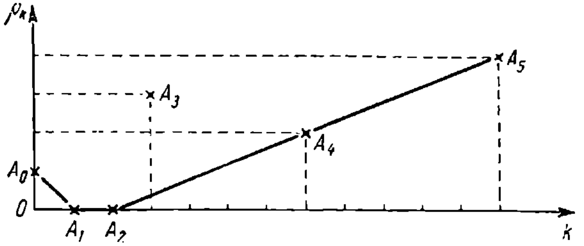

FIG. 9

segment A_0A_1) and $\varepsilon = 1/3$ (corresponding to segment A_1A_3). They correspond to the two roots of the order $x^{1/2}$ and to the three roots of the order $x^{1/3}$ found above, respectively.

EXAMPLE 3. $F(x,y) \equiv x - y + y^2 + x^3y^3 - 2x^2y^2 + x^4y^{12}$.

With the help of Newton's diagram (Fig. 9) we obtain the following values ε: $\varepsilon_1 = 1$; $\varepsilon_2 = 0$; $\varepsilon_3 = -2/5$. The equations for finding y_1, y_0 and $y_{-2/5}$ are

$$1 - y_1 = 0, \quad -y_0 + y_0^2 = 0, \quad y_{-2/5}^2 - 2y_{-2/5}^7 + y_{-2/5}^{12} = 0;$$

hence $y_1 = 1$; $y_0 = 1$; $y_{-2/5} = \sqrt[5]{1}$, while $y_{-2/5}$ is a twofold root. Therefore we have the solutions:

$$y = x + o(x), \quad y = 1 + o(1), \quad y = x^{-2/5} + o(x^{-2/5}).$$

Note 1: If $\varepsilon = \alpha/\beta$ is a fraction in its lowest terms ($\alpha \neq 0$) found by means of Newton's diagram and $\bar{y}_\varepsilon$ is a simple root of the equation (3.50), i.e.

$$\left.\frac{dP(y_\varepsilon)}{dy_\varepsilon}\right|_{y_\varepsilon = \bar{y}_\varepsilon} \neq 0,$$

then the denominators of numbers $\varepsilon, \varepsilon', \varepsilon'', \ldots,$ coincide.

This enables us to couple the method of Newton's diagram with the method of indeterminate coefficients. Namely, let $\bar{y}_\varepsilon x^\varepsilon$ be the

principal term of the expansion and $\bar{y}_\varepsilon$ be a simple root of the equation (3.50). Then y can be sought in the form

$$y = \bar{y}_\varepsilon x^\varepsilon + \sum_{i=1}^{\infty} y_{\varepsilon + i/\beta} x^{\varepsilon + i/\beta},$$

where β is the denominator of the fraction ε and $y_{\varepsilon + i/\beta}$ can be found by the method of indeterminate coefficients. In Example 3 one solution should have the form

$$y = x + \sum_{k=2}^{\infty} y_k x^k,$$

and for finding the succeeding terms of the expansion of other solutions it is necessary to apply again the method of Newton's diagram.

Note 2. The diagram can be subdivided into three sectors, the decreasing one, the constant one and the increasing one.

The decreasing sector provides the solution of the problem about implicit functions of the equation (3.35) with the condition $y(0) = 0$. The constant sector gives solutions of the problem of implicit functions of the equation (3.35) with the condition $y(0) = y_0$, where y_0 are the non-trivial solutions of the equation $F(0, y) = 0$. Finally, the increasing sector provides special solutions of the equation (3.35), for which

$$\lim_{x \to 0} y(x) = \infty.$$

2. Let us now consider the equation (3.35) in the case when $F(x, y) = \sum_{s=0}^{\infty} F_s(x) y^s$. This case differs from the preceding one only in so far as here the construction of the diagram may require an infinite number of steps. However, whatever the form of the diagram, the decreasing sector of the diagram consists of a finite number of segments and this circumstance enables us to state that the number of solutions of the equation (3.35) satisfying the condition $y(0) = 0$ is finite and gives the possibility of finding all these solutions.

All these considerations can be partially extended to include the case when y and F in (3.35) are n-dimensional vectors, i.e. when the system (3.35) coincides with the system (3.23).

The expression $F(x)y$ should then be understood as a vector-function of a vector argument, homogeneous, of order s with respect to y.

The principal term of the solution is found by the same method (provided the system (3.50) is soluble), but there is no theorem, as yet, which would be analogous to Theorem 9 for the system. It can only be stated that Note 1 holds. (The root $\bar{y}_\varepsilon$ is called simple, if the matrix $[dP(\bar{y}_\varepsilon)]/dy_\varepsilon$ is not degenerate.)

In the real domain all the reasoning holds for $x \geqslant 0$; in order to find solutions for $x < 0$ it is necessary to substitute $-x$ for x in equation (3.35).

§ 4. The Representation of Functions of *n* Variables in the Form of Superpositions

1. Suppose $y = \varphi(x)$ is a continuous function defined in the interval $[a, b]$ and mapping that interval onto the interval $[\alpha, \beta]$ and $z = f(y)$ is a continuous function defined on the interval $[\alpha, \beta]$. The variable z may be represented as a function of the variable x in the form of the combination

$$z = f[\varphi(x)], \tag{3.52}$$

which is called a *superposition of the function* $\varphi(x)$ *and* $f(y)$.

We can define similarly superpositions of functions of any number of arguments. Indeed, let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_m)$ and $Z = (z_1, z_2, \ldots, z_l)$ be points of the corresponding spaces E_n, E_m and E_l. Suppose, further, $Y = \Phi(X)$ is a function mapping some region G of space E_n into the region D of space E_m and $Z = F(Y)$ is a function defined in D. The mapping of the region G into the space E_l is carried out by means of a function, which can be written down in the form of a combination

$$Z = F[\Phi(X)], \tag{3.53}$$

which is a superposition of functions $\Phi(X)$ and $F(Y)$. Here the arguments of the functions contained in the superimposition are not connected by any relationships, so that it is possible to consider them in all kinds of combinations.

The representation of a function of several variables in the form of superpositions of functions of a smaller number of variables plays

a considerable role in nomography, and also in tabulating functions. Thus, for example, in order to tabulate a function of two variables

$$z = \ln (e^x + e^y), \tag{3.54}$$

which can be represented in the form of a superposition of functions of one variable

$$z = \chi[\varphi(x) + \psi(y)], \tag{3.55}$$

where $\varphi(t) = \psi(t) = e^t$, $\chi(u) = \ln u$, it is sufficient to have tables of functions $\varphi(t) = e^t$ and $\chi(u) = \ln u$ and there is no necessity to make up a table with two entries.

Similarly, the function of three variables

$$u = \frac{z^2 + \sin xy}{z^2 - \sin xy} \tag{3.56}$$

can be represented in the form of a superposition of functions of two variables

$$u = f[\varphi(x, y), z], \tag{3.57}$$

while

$$f(s, t) = \frac{s^2 + t}{s^2 - t}, \quad \varphi(v, w) = \sin vw. \tag{3.58}$$

Note that in the superposition (3.57), in accordance with the definition, only the functions φ and f take part, while in the superposition (3.55), in addition to the functions φ, ψ and χ, the operation of addition also occurs; the sum $\varphi(x) + \psi(y)$ plays the part of the argument of the function χ. Such superpositions are called *superpositions with addition.* They are a particular instance of superpositions with the application of arithmetical operations. Using the latter, it can be said, that the function

$$u = \frac{z^2 + \sin xy}{z^2 - \sin xy}$$

is represented in the form of superpositions of functions of one and two variables

$$u = \frac{\psi(z) + \varphi(x, y)}{\psi(z) - \varphi(x, y)},$$

where $\psi(z) = z^2$, with the arithmetical operations of addition, subtraction and division.

2. The problem of the possibility of representing functions generally by superpositions of functions of a smaller number of variables was set by David Hilbert. Naturally, an important role is played by the classes of the functions considered. For different classes, the question of the possibility of representation by superpositions is solved by different means.

The first of the theorems in this direction is Hilbert's theorem referring to the class of analytical functions. The function $f(X) = f(x_1, x_2, \ldots, x_n)$ is called an *analytical function* of n real variables in the given domain, if it belongs to class C_k for any k and if in any internal point of the domain the Taylor series converges to it. (See Chapter I, § 2.)

THEOREM 10 *(Hilbert). There exist analytical functions of three real variables which cannot be represented in the form of superpositions of analytical functions of two variables.*

Thus the "complexity" of analytical functions is characterized by the number of arguments of these functions. For functions belonging to class C_k with a fixed k, this is no longer so. The "characteristic of complexity" of functions of n variables belonging to the class C_k is the ratio n/k. This is confirmed by the following theorem.

THEOREM 11 *(Vitushkin). For any n and k there exist functions of n variables belonging to the class C_k, which cannot be represented by superpositions of functions of m variables belonging to class C_l, provided, that*

$$\frac{m}{l} < \frac{n}{k}. \tag{3.59}$$

The theorems quoted are of a negative nature and state the impossibility of representation in the form of superpositions. If we admit, however, any arbitrary continuous functions as constituents, and if we consider superpositions with addition, it does become possible to represent continuous functions in the form of such superpositions.

THEOREM 12. *Any continuous function of n variables can be represented in the form of a superposition of continuous functions of one variable with addition.*

This result was obtained through the work of V. I. Arnold and A. N. Kolmogorov. The final theorem in this field can be formulated as follows:

THEOREM 13. *For any integral $n \geqslant 2$ there exist continuous real*

functions $\psi_p^q(x)$ *defined in the unit segment* $E_1 = [0, 1]$ *such that any continuous real function* $f(x_1, x_2, \ldots, x_n)$ *defined in an n-dimensional unit cube* E_n *can be represented in the form*

$$f(x_1, x_2, \ldots, x_n) = \sum_{q=1}^{2n+1} \chi_q\left[\sum_{p=1}^{n} \psi_p^q(x)\right], \tag{3.60}$$

where the functions $\chi_q(y)$ *are real and continuous.*

In particular, any continuous function of two variables can be represented as a sum of five terms of the form

$$\chi[\varphi(x) + \psi(y)],$$

and a continuous function of three variables can be represented by a sum of seven terms of the form

$$\xi[\varphi(x) + \psi(y) + \chi(z)].$$

3. As well as expressing a function as a superposition of functions of a smaller number of variables exactly, it is sometimes important to be able to approximate a function to such superpositions. In this respect the following theorem is of the greatest interest.

THEOREM 14 *(Kolmogorov). For any* $n \geqslant 2$ *and* $\varepsilon > 0$, *there exist, for every function of n variables* $f(x_1, x_2, \ldots, x_n)$ *given and continuous in the unit cube* E_n, *polynomials*

$$b(u_1, u_2, \ldots, u_{n-1}),\ a_r(x),\ c_r(x) \quad (r = 1, 2, \ldots, n+1)$$

of $n - 1$ *variables and of one variable, such that at any point P of this cube*

$$|f(P) - \tilde{f}(P)| < \varepsilon,$$

where

$$\tilde{f}(x_1, x_2, \ldots, x_n) = \sum_{r=1,2} a_r(x_n)\, b[c_r(x_n) + x_1, \ldots, c_r(x_n) + x_{n-1}]. \tag{3.61}$$

In particular, putting, for $n = 3$

$$c_r(x) + x' = h_r(x, x'),\quad a_r(x)y = g_r(x, y),\quad u + v = d(u, v),$$

we write (3.61) in the form

$$\tilde{f}(x, y, z)$$
$$= d(g_1\{z, b[h_1(z, x), h_1(z, y)]\}, g_2\{z, b[h_2(z, x), h_2(z, h)]\}). \tag{3.62}$$

Thus, any continuous function of three variables can be approximated with any desired degree of accuracy to an expression of the form (3.62), where d, g_r, b and h_r are polynomials in two variables.

CHAPTER IV

SYSTEMS OF FUNCTIONS AND CURVILINEAR COORDINATES IN A PLANE AND IN SPACE

§ 1. Mapping. Jacobians

Mapping in the Linear, Plane and Space Cases

1. In the preceding chapter (Chapter III, § 1, secs. 1 and 2) we discussed the question of mapping by a system of n functions with n independent variables. In the present chapter we consider mappings of particular practical interest: linear, plane and spatial cases ($n = 1, 2$ and 3 respectively).

(a) *Linear case.* If $n = 1$, it is said, that a mapping of the line takes place, and it is realized by means of one function

$$u = f(x). \tag{4.1}$$

Suppose the function (4.1) is single-value and continuous in the interval l of the axis Ox. Then, to every point $p(x)$ (the original point or inverse image) of this interval, this function refers a unique point $q(u)$ (mapping or image) of the interval λ on the axis $O'u$, which is the mapping (image) of the interval l, which, in turn, serves as its original (inverse image).

If the function

$$x = \varphi(u) \tag{4.2}$$

is the inverse of the function (4.1), it is said that it gives *an inverse mapping* with respect to the one defined by the function (4.1). The inverse map (4.2) maps the points of the interval λ onto the points of the interval l; now, λ serves as the original (inverse image) for the map (image) l.

The inverse mapping may be many-valued.

(b) *The plane case.* When $n = 2$, *the plane case* of mapping takes place. It is realized in this case by the system of functions

$$\left.\begin{aligned} u &= f(x, y), \\ v &= g(x, y). \end{aligned}\right\} \tag{4.3}$$

Suppose the functions f and g are single-valued and continuous in the region D of a plane which has a system of cartesian coordinates Oxy. Then, to every point $p(x, y)$ of the region D, the system (4.3) assigns a unique point $q(u, v)$ of a region Δ of a plane which has a system of cartesian coordinates $O'uv$. The region Δ is the mapping (image) of the region D, which serves as its original (inverse image).

If the system (4.3) is solved with respect to x and y and a new system is obtained

$$\left.\begin{aligned} x &= \varphi(u, v), \\ y &= \psi(u, v), \end{aligned}\right\} \tag{4.4}$$

it is said that the system (4.4) gives the *inverse* mapping with respect to the one defined by system (4.3). The inverse mapping will not necessarily be single-valued.

(c) *The space case.* When $n = 3$ we have the space mapping, realizable by the system of functions

$$\left.\begin{aligned} u &= f(x, y, z), \\ v &= g(x, y, z), \\ w &= h(x, y, z). \end{aligned}\right\} \tag{4.5}$$

Suppose functions f, g and h are single-valued and continuous in the region Q of the space provided by the system of cartesian coordinates $Oxyz$. Then to every point $p(x, y, z)$ of the region Q, the system (4.5) assigns a unique point $q(u, v, w)$ of the region Ω of the space provided by the system of cartesian coordinates $O'uvw$. The region Ω is the mapping (image) of the region Q, which serves as the original (inverse image) of Ω.

If the system (4.5) is solved with respect to x, y and z, it is said that the newly obtained system

$$\left.\begin{aligned} x &= \varphi(u, v, w), \\ y &= \psi(u, v, w), \\ z &= \chi(u, v, w) \end{aligned}\right\} \tag{4.6}$$

gives the *inverse* mapping of that defined by the system (4.5). The inverse map may be many-valued.

Homeomorphic Mapping

2. If (for every n) not only the given (direct) mapping but its inverse are single-valued, the given mapping is called *bi-unique*. If all functions involved both in the given and the inverse mappings are continuous we say that the given mapping is *bi-continuous*.

A mapping that is bi-unique and bi-continuous is called *homeomorphic*. A homeomorphic mapping possesses the property that to every point of the inverse image it assigns a unique point of the image, and to any two distinct points of the inverse image it assigns two distinct points of the image.

Affine Mapping

3. If every function of a system which is involved in the mapping (for any n) is a *linear* function, the corresponding map is called *affine*.

(a) *The one-dimensional case.* The affine mapping in the one-dimensional case is defined by means of the function

$$u = ax + b. \tag{4.7}$$

If $a \neq 0$, the mapping (4.7) is *invertible:*

$$x = \frac{1}{a}u - \frac{1}{a}b. \tag{4.8}$$

The *inverse* mapping (4.8) is also affine. From (4.7) and (4.8) we note that u as a function of x and, conversely, x as a function of u are single-valued and continuous everywhere. Therefore, when $a \neq 0$, the affine mapping is homeomorphic everywhere. Suppose the interval $[x_1, x_2]$ of the axis Ox is mapped into the segment $[u_1, u_2]$ of the axis $O'u$ by means of the function (4.7). Then

$$a = \frac{u_2 - u_1}{x_2 - x_1}$$

and therefore $|a|$ expresses the ratio of the length of the image-interval to the length of the original interval. For this reason $|a|$ is

called *the coefficient of distortion* (of length) in the affine mapping (4.7).

The sign of a provides the direction of displacement of the point mapped: if $a > 0$, the image point moves down the interval $[u_1, u_2]$ in the same direction as the original point moves down the interval $[x_1, x_2]$; if $a < 0$, the directions of displacement of the image point and the original point are opposite.

If $a = 0$, the mapping (4.7) is not invertible and therefore, of course, not homeomorphic. Here the function (4.7) is written down in the form $u = b$. It maps the whole of the Ox axis into one point b on the axis $O'u$, covered an infinite number of times. In the case of $a = 0$ the mapping (4.7) is called *degenerate*.

If, on the axis Ox, we take *uniformly distributed* points (i.e. the distance between each pair of neighbouring points is constant) then a non-degenerate ($a \neq 0$) affine mapping (4.7) maps them into points of the axis $O'n$, also uniformly distributed.

(b) *The plane case*. In the plane case, an affine mapping is realized by means of a set of functions

$$\left.\begin{aligned} u &= a_1x + b_1y + c_1, \\ v &= a_2x + b_2y + c_2. \end{aligned}\right\} \tag{4.9}$$

If the determinant of the system is non-zero:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0, \tag{4.10}$$

the set (4.9) can be solved with respect to x and y:

$$\left.\begin{aligned} x &= a'_1u + b'_1v + c'_1, \\ y &= a'_2u + b'_2v + c'_2, \end{aligned}\right\} \tag{4.11}$$

and this shows, that the inverse of an affine mapping is also an affine mapping. Therefore, in the fulfilment of the condition (4.10), the affine mapping (4.9) turns out to be homeomorphic.

If the condition (4.10) is not fulfilled then the mapping (4.9), having become non-invertible, ceases to be homeomorphic and is now called *degenerate;* it can map the whole plane Oxy into a straight line or into a point of the plane $O'uv$.

The basic properties of affine mappings are briefly summarized below.

1°. If two systems of straight lines are constructed in a plane, each system consisting of straight lines parallel and equidistant from each other, the plane is cut up into regions bounded by equal parallelograms. It is said that points at the vertices of these parallelograms are *uniformly distributed* in the plane.

It turns out, that in carrying out the condition (4.10), the affine mapping (4.9) transforms a system of points uniformly distributed in the plane Oxy into a system of points uniformly distributed in the plane $O'uv$. No other mapping of form (4.3) possesses this property.

2°. The expression *the algebraic magnitude of the area* of a region is understood to mean its area taken with the sign "+" if the contour of the region is traced out in the positive direction (i.e. so that the region itself remains on the left), and taken with the sign "−" if the tracing is done in the negative direction (i.e. in the direction opposite to the indicated one).

We denote the algebraic magnitudes of the image-area (in the plane $O'uv$) and of the inverse-image-area (in the plane Oxy) by $\Delta\delta$ and Δs, and the modulus of their ratio by k,

$$k = \left|\frac{\Delta\sigma}{\Delta s}\right|. \tag{4.12}$$

It turns out, that for a non-degenerate affine mapping (4.9) this *coefficient of distortion* (of area) k is a *constant* independent of the shape of the region mapped, and equal to the *modulus of the determinant* of the mapping (4.9),

$$k = |a_1 b_2 - a_2 b_1|. \tag{4.13}$$

The sign of the determinant of the mapping (4.9) can also be interpreted in a visual manner: if $a_1 b_2 - a_2 b_1 > 0$, this means that to the displacement of the inverse-image-point along the contour of the region being mapped in any definite direction (positive or negative), there corresponds the displacement of the image-point along the contour of the mapped region in *the same* direction; if $a_1 b_2 - a_2 b_1 < 0$, the image-point moves in a direction opposite to the direction of the movement of the inverse-image point.

3°. An important particular case of affine mapping are affine mappings *preserving distances between points*. Such mappings preserve also the *shape* of the region and so its *area*. Hence it follows

that affine mappings of this kind should be sought among mappings of the form (4.9) in which

$$k = |a_1 b_2 - a_2 b_1| = 1. \tag{4.14}$$

It turns out that those and only those affine mappings *preserve distances* which can be represented in one of the following *four* forms:

$$\left.\begin{aligned} u &= x\cos\varphi \mp y\sin\varphi + c_1, \\ v &= \pm x\sin\varphi + y\cos\varphi + c_2 \end{aligned}\right\} \tag{4.15}$$

and

$$\left.\begin{aligned} u &= x\cos\varphi \mp y\sin\varphi + c_1, \\ v &= \mp x\sin\varphi - y\cos\varphi + c_2, \end{aligned}\right\} \tag{4.16}$$

where φ is an arbitrary angle.

The equations (4.15) are the formulae for the transformation of coordinates in a plane: the parallel transfer of the coordinate axes with a new origin at the point $O'(c_1, c_2)$ and the rotation of the system of coordinates either through the angle φ (for upper signs) or through the angle $-\varphi$ (for lower signs). Therefore, in a mapping of form (4.15) the image of a given region is obtained by transferring it as a rigid body—first by rotation about the origin of the coordinate axes, and then by shifting it parallel to itself. Therefore, the affine mapping of form (4.15) is called *the plane motion of a rigid lamina* (region) or simply rigid *motion*. (The mappings of form (4.16) cannot be called motion in any sense, since they demand also the symmetrical reflection of the region with respect to a coordinate axis, which cannot be achieved by mere motion in a plane.)

4°. The following belongs to a type of affine mapping commonly encountered:

$$\left.\begin{aligned} u &= ax, \\ v &= ay, \quad a \neq 0. \end{aligned}\right\} \tag{4.17}$$

This defines the transformation of similarity with centre at the point $O(0, 0)$. The number a is called the *coefficient of similarity*. Suppose the plane $O'uv$ coincides with the plane Oxy. Then, in the case $a > 0$, the point $p(x, y)$ maps into the point $q(u, v)$ lying on the same ray as the point $p(x, y)$ itself, and in the case $a < 0$ it maps into a point lying on the opposite ray, while in both these cases the ratio of segments Oq and Op (the distances of the image and the inverse image from the centre of similarity) remains equal $|a|$. The

transformation of similarity maps any figure (region) into a figure (region) similar to the first one, with a coefficient of similarity $|a|$.

(c) *The space case.* The affine mapping in the space case is realized by the system

$$\left.\begin{aligned} u &= a_1x + b_1y + c_1z + d_1, \\ v &= a_2x + b_2y + c_2z + d_2, \\ w &= a_3x + b_3y + c_3z + d_3. \end{aligned}\right\} \qquad (4.18)$$

If we assume that the determinant of the system (4.18) differs from zero:

$$\delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0, \qquad (4.19)$$

then the system (4.18) can be solved with respect to x, y and z:

$$\left.\begin{aligned} x &= a_1'u + b_1'v + c_1'w + d_1', \\ y &= a_2'u + b_2'v + c_2'w + d_2', \\ z &= a_3'u + b_3'v + c_3'w + d_3', \end{aligned}\right\} \qquad (4.20)$$

and this shows that a mapping inverse to an affine one is also affine mapping. Therefore, on fulfilment of condition (4.19), the affine mapping (4.18) turns out to be homeomorphic.

If the condition (4.19) is not fulfilled, then the mapping (4.18), having become non-invertible, ceases being homeomorphic and is called *degenerate:* it may map the whole of the space $Oxyz$ into one plane of the space $Ouvw$.

If we construct in space three systems of planes each consisting of planes which are parallel and equidistant from each other, the space is cut up into regions bounded by equal parallelepipeds. It is said that the points at the vertices are *uniformly distributed* in space.

It turns out that the non-degenerate affine mapping (4.18) transforms a system of points uniformly distributed in the space $Oxyz$ into a system of points uniformly distributed in the space $O'uvw$. No other mapping of form (4.5) possesses this property.

For the non-degenerate affine mapping (4.18), *the coefficient of*

distortion (of volume) k, i.e. the ratio of the volume of the image-region to the volume of the inverse-image region, is a *constant* independent of the shape of the region and equal to *the modulus of the determinant* of the system (4.18):

$$k = |\delta|. \tag{4.21}$$

4. Coefficient of Distortion: Jacobian

The constancy of the coefficient of distortion (of length, of area, of volume) at all points (of an axis, of a plane, of space) holds for non-degenerate affine mappings and only for those. For non-affine mappings there arises the necessity of determining *locally* the coefficient of distortion.

(a) *The linear case.* The name coefficient of distortion $k(x_0)$ at the point $p_0(x_0)$ of the mapping (4.1), homeomorphic at point p_0, is given to the *limit of the ratio* of the length of the mapped interval (on the axis $O'u$) to the length of the interval being mapped (on the axis Ox) beginning at point p_0, when that interval contracts to the point p_0.

On the assumption of the *differentiability* of the function $f(x)$ it follows from this definition that the coefficient of distortion equals *the modulus of the derivative* of the mapping function, taken at the given point

$$k(x_0) = |f'(x_0)|. \tag{4.22}$$

(b) *The plane case.* The name *coefficient of distortion* $k(x_0, y_0)$ at the point $p_0(x_0, y_0)$ of the mapping (4.3), homeomorphic at point p_0, is given to *the limit of the ratio* of the area of the mapped region (in the plane $O'uv$) to the area of a rectangle being mapped (in the plane Oxy) with a vertex at the point p_0 and with sides parallel to the axes Ox and Oy, when this rectangle contracts unboundedly to the point p_0.

On assuming the differentiability of the function $f(x, y)$ and $g(x, y)$ it follows from the definition that the coefficient of distortion equals *the modulus of the Jacobian* of the mapping (4.3):

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}. \tag{4.23}$$

Thus

$$k(x_0, y_0) = \left| \frac{\partial(u, v)}{\partial(x, y)} \right|_{\substack{x=x_0 \\ y=y_0}}. \tag{4.24}$$

(c) *The space case.* The name *coefficient of distortion* $k(x_0, y_0, z_0)$ at the point $p_0(x_0, y_0, z_0)$ of the mapping (4.5), homeomorphic at point p_0, is given to *the limit of the ratio* of the volume of the mapped region (in the space $O'uvw$) to the volume of a rectangular parallelepiped being mapped (in the space $Oxyz$) with a vertex at point p_0 and with faces parallel to the planes xOy, yOz and zOx, when that parallelepiped contracts unboundedly into the point p_0.

If the differentiability of the function $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$ is assumed, it follows from the definition that the coefficient of distortion equals *the modulus of the Jacobian* of the mapping (4.5):

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}. \tag{4.25}$$

Thus,

$$k(x_0, y_0, z_0) = \left| \frac{\partial(u, v, w)}{\partial(x, y, z)} \right|_{\substack{x=x_0 \\ y=y_0 \\ z=z_0}}. \tag{4.26}$$

Conformal Mapping

5. Suppose, the functions u and v, which are defined by the mapping (4.3), are *harmonic*, i.e. suppose they are twice continuously differentiable and that each satisfies *Laplace's equation:*

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0, \end{aligned}\right\} \tag{4.27}$$

and suppose they are connected by means of one of the following relationships:

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\right\} \tag{4.28}$$

or

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}.\end{aligned}\right\} \tag{4.29}$$

Then, in the case of fulfilment of condition (4.28), the mapping (4.3) is called *conformal* (or *regular*) *of the first kind*, and in the case of fulfilment of condition (4.29) it is called a *conformal mapping of the second kind.*

Conformal mappings possess the property that they map any infinitely small region of the plane Oxy into an infinitely small region of the plane $O'uv$, similar to the first one. Here the coefficient of similarity equals the square root of the Jacobian of the mapping (it is assumed that the latter does not equal zero). Often these geometric properties of conformal mapping are used as a basis of its definition.†

Note: In a three-dimensional case (and even more so, when $n > 3$) a conformal mapping can be reduced to transformation of shift, similarity and inversion only *(Liouville's theorem).*

§ 2. Curvilinear Coordinates in a Plane

1. Let us consider the mapping

$$\left.\begin{aligned}u &= f(x, y),\\ v &= g(x, y),\end{aligned}\right\} \tag{4.30}$$

† Conformal mapping is often understood to mean only mapping with a one-to-one correspondence. Sometimes a distinction is made between the ideas of conformity and of regularity of mapping, calling only those mappings conformal that are in one-to-one correspondence.

homeomorphic in some region D of the plane Oxy, and the mapping inverse to it,

$$\left.\begin{aligned} x &= \varphi(u, v), \\ y &= \psi(u, v), \end{aligned}\right\} \tag{4.31}$$

which is also homeomorphic in the region Δ of the plane $O'uv$ (Δ is the image of the region D).

The curvilinear coordinates of point p, belonging to the region D and having as its rectangular cartesian coordinates the numbers x and y, are the numbers u and v, which serve as rectangular cartesian coordinates of the point q in the region Δ, which is the image of the point p for a given homeomorphic mapping.

These numbers u, v can be called *coordinates* of the point $p(x, y)$ since they are determined uniquely from chosen x, y (from a chosen point p) by means of the system (4.30), and, conversely, from any chosen pair of numbers u and v and with the help of the system (4.31) the pair of numbers x, y and so the unambiguous point $p(x, y)$ is determined uniquely.

The set of points of the region D, one of whose curvilinear coordinates is constant ($u = u_0 = \text{const}$ or $v = v_0 = \text{const}$) is called the corresponding *coordinate line* in the given system of curvilinear coordinates. The coordinate lines $u = u_0$ and $v = v_0$ in the plane Oxy are defined by the equations

$$f(x, y) = u_0 \tag{4.32}$$

and

$$g(x, y) = v_0 \tag{4.33}$$

respectively.

For various values of u_0 and v_0 (however, such that the point p_0 defined by them is never outside the region D) *two systems* of coordinate lines are formed covering the region D and subdividing it into curved quadrilaterals (in the case of affine mapping, the latter have the form of parallelograms).

Systems of Orthogonal Curvilinear Coordinates in a Plane. Conditions of Orthogonality

2. Let us denote sets of coordinate lines of form $u = u_0$ and of form $v = v_0$ by U and V respectively.

If any two coordinate lines, one of which is taken from the set U and the other from the set V, intersect at right angles, then the

system of curvilinear coordinates possessing this property is called *rectangular* or *orthogonal.*

The mapping (4.30) defines an orthogonal system of curvilinear coordinates if

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=0. \tag{4.34}$$

This condition of orthogonality of the system is written down in the following form:

$$S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right)=0, \tag{4.35}$$

where the symbol S means *the sum* of two terms: the expression within the brackets and the expression, analogous to the above, obtainable from the first one by exchanging x for y.

It is possible to obtain also another condition of orthogonality:

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}=0 \tag{4.36}$$

or

$$S\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right)=0. \tag{4.37}$$

Any conformal mapping (see § 1, sec. 5) defines a system of orthogonal curvilinear coordinates. The converse proposition does not hold, however.

Lamé Coefficients and Differential Parameters of the First Order of a System of Curvilinear Coordinates in a Plane

3. In various applications of curvilinear coordinates so-called *Lamé coefficients l_u and l_v and differential parameters of the first order h_u and h_v*, defined by the following formulae, are used:

$$\left.\begin{aligned} l_u &= \sqrt{S\left(\frac{\partial x}{\partial u}\right)^2}, \\ l_v &= \sqrt{S\left(\frac{\partial x}{\partial v}\right)^2}, \end{aligned}\right\} \tag{4.38}$$

$$\left.\begin{aligned} h_u &= \sqrt{S\left(\frac{\partial u}{\partial x}\right)^2}, \\ h_v &= \sqrt{S\left(\frac{\partial v}{\partial x}\right)^2}. \end{aligned}\right\} \tag{4.39}$$

For the curvilinear coordinates of an orthogonal system, Lamé coefficients are inverse in magnitude to the corresponding differential parameters of the first order, i.e.

$$l_u = \frac{1}{h_u}, \quad l_v = \frac{1}{h_v}.$$

An Element of Length and an Element of Area in a System of Curvilinear Coordinates in a Plane

4. For the element ds of the length of a curve in the system (4.30) of orthognal curvilinear coordinates the following formula holds:

$$ds = \sqrt{l_u^2\, du^2 + l_v^2\, dv^2} = \sqrt{\frac{1}{h_u^2}\, du^2 + \frac{1}{h_v^2}\, dv^2}. \tag{4.40}$$

The expressions for the elements ds_u and ds_v of the length of the coordinate lines $u = u_0$ and $v = v_0$ are as follows:

$$ds_u = l_u du, \quad ds_v = l_v dv. \tag{4.41}$$

For *the element dq of the area of a region* in the system of orthogonal curvilinear coordinates in a plane the formula

$$dq = l_u l_v\, du\, dv = \frac{1}{h_u h_v}\, du\, dv \tag{4.42}$$

holds.

On the other hand, it is known that the coefficient of distortion of area in mapping is the modulus of the Jacobian (see § 1). Denoting the Jacobian of mapping (4.31) by J, i.e. putting

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}, \tag{4.43}$$

we have that

$$dq = |J|\, du\, dv. \tag{4.44}$$

Hence it follows that for a system of orthogonal curvilinear coordinates

$$|J| = l_u l_v = \frac{1}{h_u h_v}. \tag{4.45}$$

Some Systems of Curvilinear Coordinates in a Plane

5. Below, we describe some widely used systems of curvilinear coordinates in a plane; we proceed according to the following scheme: we give (a) their definition (i.e. the expression of rectangular cartesian coordinates x and y in terms of the curvilinear coordinates u and v), (b) coordinate lines, (c) formulae for Lamé coefficients l_u and l_v (the differential parameters of the first order h_u and h_v for orthogonal systems are calculated as the reciprocals of their values), (d) the formula for the element ds of length, and (e) the formula for the element dq of area.

1°. CARTESIAN COORDINATES

(a)

$$\left.\begin{aligned} x &= a_1'u + b_1'v + c_1', \\ y &= a_2'u + b_2'v + c_2', \end{aligned}\right\} \tag{4.46}$$

where the constant magnitudes a', b', c' are selected in such a way that the Jacobian $J_1 = \partial(x, y)/\partial(u, v)$ is non-zero.

The system (4.46) is transformed into the following

$$\left.\begin{aligned} u &= a_1x + b_1y + c_1, \\ v &= a_2x + b_2y + c_2 \end{aligned}\right\} \tag{4.47}$$

(a, b, c constant), and the Jacobian $J = \partial(u, v)/(\partial(x, y)$ of the transformed system is also non-zero; it is reciprocal in value to

$$J = \frac{1}{J_1}.$$

(b) The coordinate lines are straight lines parallel to the straight lines

$$\left.\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0 \end{aligned}\right\} \tag{4.48}$$

and intersecting at an angle ω for which $\tan\omega \dfrac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}$.

Since

$$S\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\right) = a_1 a_2 + b_1 b_2, \tag{4.49}$$

the cartesian system is, generally speaking, non-orthogonal. It is orthogonal only in the case when $a_1 a_2 + b_1 b_2 = 0$; here $\omega = \pi/2$. Otherwise, it is called *oblique*.

The condition of equality of scale in the systems Oxy and $O'uv$ is that the equations

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = a_1'^2 + a_2'^2 = b_1'^2 + b_2'^2 = 1.$$

(c)

$$\left.\begin{aligned} l_u &= \sqrt{a_1'^2 + a_2'^2} = \frac{1}{\sqrt{a_1^2 + b_1^2}}, \\ l_v &= \sqrt{b_1'^2 + b_2'^2} = \frac{1}{\sqrt{a_2^2 + b_2^2}} \end{aligned}\right\} \tag{4.50}$$

must be satisfied.

For unchanging scales $l_u = l_v = 1$.

(d)

$$\begin{aligned} ds &= \sqrt{(a_1'^2 + a_2'^2)\,du^2 + (b_1'^2 + b_2'^2)\,dv^2} \\ &= \sqrt{\frac{du^2}{a_1^2 + b_1^2} + \frac{dv^2}{a_2^2 + b_2^2}}. \end{aligned} \tag{4.51}$$

For unchanging scales $ds = \sqrt{du^2 + dv^2}$.

(e)

$$\begin{aligned} dq &= \sqrt{(a_1'^2 + a_2'^2)(b_1'^2 + b_2'^2)}\,du\,dv \\ &= \frac{1}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}\,du\,dv. \end{aligned} \tag{4.52}$$

For unchanging scales $dq = du\,dv$.

2°. Polar Coordinates

(a)

$$\left.\begin{aligned} x &= u\cos v, \\ y &= u\sin v, \end{aligned}\right\} \tag{4.53}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi.$$

It is customary to use different notation in polar coordinates: $u \equiv \varrho$, $v \equiv \varphi$; *ϱ is the polar radius, φ is the polar angle.*

The system is orthogonal.

At the point (0, 0) the Jacobian of the polar system of coordinates equals zero. Therefore, the appropriate mapping in it is no longer homeomorphic; in the given point the coordinate u equals zero, and the coordinate v can take any value in the semi-open interval $[0, 2\pi]$. The origin of the coordinates is *a special point* for the system of polar coordinates.

(b) Coordinate lines are concentric circles

$$x^2 + y^2 = u_0^2 \tag{4.54}$$

with centre at the origin and radius u_0 and rays

$$x = u \cos v_0, \quad y = u \sin v_0, \tag{4.55}$$

issuing from the origin and inclined to the axis Ox at an angle v_0. The parametric equations of these rays, shown above, can be written down, in the case of $v_0 \neq \pi/2$, $3\pi/2$, in the form

$$y = (\tan v_0)x. \tag{4.56}$$

(c)
$$\left.\begin{aligned} l_u &= 1, \\ l_v &= u. \end{aligned}\right\} \tag{4.57}$$

(d)
$$ds = \sqrt{du^2 = u^2\, dv^2}\,. \tag{4.58}$$

(e)
$$dq = u\, du\, dv. \tag{4.59}$$

3°. Generalized Polar Coordinates

(a)
$$\left.\begin{aligned} x &= au \cos v, \\ y &= bu \sin v, \end{aligned}\right\} \tag{4.60}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad a > 0, \quad b > 0, \quad a \neq b.$$

The point (0, 0) is a special point of the system of generalized polar coordinates.

The system is not orthogonal (it is orthogonal only in the case $a = b$).

(b) Coordinate lines are concentric ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = u_0^2 \tag{4.61}$$

with centre at the origin and with semi-axes au_0, bu_0, and the rays

$$x = au\cos v_0, \quad y = bu\sin v_0, \tag{4.62}$$

issuing from the origin and with slope $b/a \tan v_0$.

(c)
$$\left.\begin{aligned} l_u &= \sqrt{a^2\cos^2 v + b^2\sin^2 v},\\ l_v &= u\sqrt{a^2\sin^2 v + b^2\cos^2 v};\end{aligned}\right\} \tag{4.63}$$

$$\left.\begin{aligned} h_u &= \sqrt{\frac{\cos^2 v}{a^2} + \frac{\sin^2 v}{b^2}},\\ h_v &= \frac{1}{u}\sqrt{\frac{\sin^2 v}{a^2} + \frac{\cos^2 v}{b^2}}.\end{aligned}\right\} \tag{4.64}$$

When $a \neq b$ in this case $l_u h_u \neq 1$ and $l_v h_v \neq 1$, since here the system of generalized polar coordinates is not orthogonal.

(d)
$$ds = [(a^2\cos^2 v + b^2\sin^2 v)\,du^2 + (b^2 - a^2)u\sin 2v\,du\,dv + (a^2\sin^2 v + b^2\cos^2 v)u^2\,dv^2]^{1/2}. \tag{4.65}$$

(e)
$$dq = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = abu\,du\,dv. \tag{4.66}$$

4°. General Elliptical Coordinates

(a)
$$\left.\begin{aligned} x^2 &= \frac{(u + a^2)(v + a^2)}{a^2 - b^2},\\ y^2 &= \frac{(u + b^2)(v + b^2)}{b^2 - a^2},\end{aligned}\right\} \tag{4.67}$$

$$0 \leqslant b^2 < a^2, \quad -a^2 < v < -b^2, \quad -b^2 < u < +\infty.$$

To every pair of coordinates u and v there correspond four points (x, y) (one in each quadrant) symmetrical with respect to the coordinate axes. Therefore, for homeomorphism it is necessary to limit oneself to one quadrant of the plane Oxy, for example, the first one: $x > 0, y > 0$. This whole quadrant is a region of homeomorphism of a system of elliptical coordinates.

The system is orthogonal.

(b) The coordinate lines are confocal ellipses

$$\frac{x^2}{u_0 + a^2} + \frac{y^2}{u_0 + b^2} = 1 \tag{4.68}$$

and hyperbolae

$$\frac{x^2}{v_0 + a^2} - \frac{y^2}{-(v_0 + b^2)} = 1, \quad -(v_0 + b^2) > 0, \tag{4.69}$$

with foci at points $F_1(-\sqrt{a^2 - b^2}, 0)$ and $F_2(\sqrt{a^2 - b^2}, 0)$.

(c)
$$\left.\begin{aligned} l_u^2 &= \frac{1}{4}\frac{u - v}{m(u)}, \\ l_v^2 &= \frac{1}{4}\frac{u - v}{-m(v)}, \end{aligned}\right\} \tag{4.70}$$

where $m(t) = (t + a^2)(t^2 + b^2)$. Note that $m(v) < 0$ for all admissible values v.

(d)
$$ds = \frac{1}{2}\sqrt{\frac{u - v}{m(u)}\,du^2 + \frac{u - v}{-m(v)}\,dv^2}\,. \tag{4.71}$$

(e)
$$dq = \frac{u - v}{4\sqrt{-m(u)\,m(v)}}\,du\,dv. \tag{4.72}$$

5°. Degenerate Elliptical Coordinates

(a)
$$\left.\begin{aligned} x &= \cosh u \cos v, \\ y &= \sinh u \sin v, \end{aligned}\right\} \tag{4.73}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi.$$

Degenerate elliptical coordinates represent a particular case of the general elliptical coordinates which arises when $a = 1$ and $b = 0$ and on substituting $\sinh^2 u$ and $-\sin^2 v$ for u and v respectively.

The system of equations determining degenerate elliptical coordinates maps homeomorphically the whole region Oxy from which the interval $(-1, +\infty)$ of the axis Ox has been removed, into a semi-infinite strip of the plane Ouv, bounded by the three lines

$$v = 0,\ u \geqslant 0; \quad v = 2\pi,\ u \geqslant 0; \quad u = 0,\ v \geqslant 0. \tag{4.74}$$

The degenerate elliptic coordinates $\bar{u}$ and $\bar{v}$ and the general elliptic coordinates u and v are connected with each other in the following manner:

$$u = \int_{-b^2}^{\bar{u}} \frac{dt}{\sqrt{4m(t)}}, \quad v = \int_{-b^2}^{\bar{v}} \frac{dt}{\sqrt{-4m(t)}}, \tag{4.75}$$

where the function $m(t)$ is of the same form as above.

The system is orthogonal.

(b) The coordinate lines are confocal ellipses

$$\frac{x^2}{\cosh^2 u_0} + \frac{y^2}{\sinh^2 u_0} = 1, \quad u_0 \neq 0, \tag{4.76}$$

and hyperbolae

$$\frac{x^2}{\cos^2 v_0} - \frac{y^2}{\sin^2 v_0} = 1, \quad v_0 \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \tag{4.77}$$

with foci at points $F_1(-1, 0)$ and $F_2(1, 0)$.

(c) $$l_u = l_v = \sqrt{\cosh^2 u - \cos^2 v}. \tag{4.78}$$

(d) $$ds = \sqrt{\cosh^2 u - \cos^2 v}\,\sqrt{du^2 + dv^2}. \tag{4.79}$$

(e) $$dq = (\cosh^2 u - \cos^2 v)\, du\, dv. \tag{4.80}$$

6⁰. Parabolic Coordinates

(a) $$\left.\begin{aligned} x &= u^2 - v^2, \\ y &= 2uv, \end{aligned}\right\} \tag{4.81}$$

$$-\infty < u < +\infty, \quad 0 \leqslant v < +\infty.$$

The system of equations defining parabolic coordinates maps homeomorphically the whole plane Oxy, from which the positive semi-axis Ox has been removed, into the upper half-plane $v > 0$ of the plane Ouv. This mapping is conformal (see § 1, sec. 5).

The system is orthogonal.

(b) The coordinate lines are parabolae

$$x = u_0^2 - \frac{y^2}{4u_0^2} \tag{4.82}$$

and

$$x = \frac{y^2}{4v_0^2} - v_0^2 \tag{4.83}$$

with parameters $2u_0^2$ and $2v_0^2$ respectively. The axis of the first of the

parabolae is the ray $(-\infty, u_0^2)$ and of the second is the ray $(-v_0^2, +\infty)$.

(c) $$l_u = l_v = 2\sqrt{u^2 + v^2}. \tag{4.84}$$

(d) $$ds = 2\sqrt{u^2 + v^2}\sqrt{du^2 + dv^2}. \tag{4.85}$$

(e) $$dq = 4(u^2 + v^2)\, du\, dv. \tag{4.86}$$

7°. Bipolar Coordinates

(a) $$\left.\begin{aligned} x &= \frac{\sinh u}{\cosh u + \cos v}, \\ y &= \frac{\sin v}{\cosh u + \cos v}, \end{aligned}\right\} \tag{4.87}$$

$$-\infty < u < +\infty, \quad 0 \leqslant v < 2\pi.$$

The system defining bipolar coordinates maps homeomorphically the whole plane Oxy, from which the segment $[-1, 1]$ of the axis Ox has been removed, into an infinite strip of the plane Ouv, bounded by the straight lines $v = 0$ and $v = 2\pi$. This mapping is conformal (see § 1, sec. 5).

The system is orthogonal.

(b) The coordinate lines are circles

$$(x - \coth u_0)^2 + y^2 = \frac{1}{\sinh^2 u_0} \tag{4.88}$$

and

$$x^2 + (y + \cot v_0)^2 = \frac{1}{\sin^2 v_0} \tag{4.89}$$

of radii $\dfrac{1}{|\sinh u_0|}$ and $\dfrac{1}{|\sin v_0|}$ respectively. The centre of the first one lies at the point $(\coth u_0, 0)$ and of the second one at the point $(0, -\cot v_0)$.

(c) $$l_v = l_u = \frac{1}{\cosh u + \cos v}. \tag{4.90}$$

(d) $$ds = \frac{1}{\cosh u + \cos v}\sqrt{du^2 + dv^2}. \tag{4.91}$$

(e) $$dq = \frac{1}{(\cosh u + \cos v)^2}\, du\, dv. \tag{4.92}$$

§ 3. Curvilinear Coordinates in Space

1. The name *curvilinear coordinates* of the point p, whose rectangular cartesian coordinates are the numbers x, y and z is applied to the rectangular cartesian coordinates u, v and w of point q, which serves as the image of the point p in the mapping

$$\left.\begin{aligned} u &= f(x, y, z), \\ v &= g(x, y, z), \\ w &= h(x, y, z), \end{aligned}\right\} \tag{4.93}$$

homeomorphic in some region Q containing point p.

The set of points of region Q, one of whose curvilinear coordinates is constant ($u = u_0$ = const, or $v = v_0$ = const, or $w = w_0$ = const), is called a *coordinate surface* in the given system of coordinates, and the set of points of region Q, two of whose curvilinear coordinates are constant ($u = u_0$ and $v = v_0$ or $u = u_0$ and $w = w_0$ or $v = v_0$ and $w = w_0$), is called *a coordinate line* in the given system of coordinates.

The coordinate surfaces $u = u_0$, $v = v_0$, $w = w_0$ in the space $Oxyz$ are defined by equations

$$f(x, y, z) = u_0, \quad g(x, y, z) = v_0, \quad h(x, y, z) = w_0. \tag{4.94}$$

Coordinate lines are formed at the intersection of pairs of coordinate surfaces and are defined by the sets of equations

$$\left.\begin{aligned} f(x, y, z) &= u_0, \\ g(x, y, z) &= v_0, \end{aligned}\right\} \quad \left.\begin{aligned} f(x, y, z) &= u_0, \\ h(x, y, z) &= w_0, \end{aligned}\right\} \quad \left.\begin{aligned} g(x, y, z) &= v_0, \\ h(x, y, z) &= w_0. \end{aligned}\right\} \tag{4.95}$$

For various u_0, v_0, w_0 (such, however, that the point which they define does not leave region Q) three systems of coordinate surfaces form a grid covering the region Q and subdividing it into curved hexahedra (in the case of affine mapping the latter take the form of parallelepipeds). Let us denote sets of coordinate surfaces of form $u = u_0$, of form $v = v_0$ and of form $w = w_0$ by U, V and W respectively. If any three coordinate surfaces, of which one is taken from the set U, another from the set V and the third from the set W, intersect in pairs at right angles with each other, the system of

curvilinear coordinates with this property is called *rectangular* or *orthogonal.*

The condition of orthogonality of a system of coordinates is the simultaneous fulfilment of the equations

$$\mathcal{S}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right)=0,\quad \mathcal{S}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right)=0,\quad \mathcal{S}\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\right)=0, \tag{4.96}$$

in which the symbol $\mathcal{S}$ denotes the sum of *three* terms: of the expression in brackets and similar ones obtained in substituting first y for x, then z for x in it.

The condition of orthogonality can be expressed otherwise in the following form:

$$\mathcal{S}\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right)=0,\quad \mathcal{S}\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial w}\right)=0,\quad \mathcal{S}\left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial w}\right)=0 \tag{4.97}$$

(the symbol $\mathcal{S}$ has the same meaning as before).

The *Lamé coefficients* L_u, L_v and L_w *and the differential parameters of the first order* H_u, H_v, H_w for a system of curvilinear coordinates in space are expressed as follows:

$$L_u=\sqrt{\mathcal{S}\left(\frac{\partial x}{\partial u}\right)^2},\quad L_v=\sqrt{\mathcal{S}\left(\frac{\partial x}{\partial v}\right)^2},\quad L_w=\sqrt{\mathcal{S}\left(\frac{\partial x}{\partial w}\right)^2}; \tag{4.98}$$

$$H_u=\sqrt{\mathcal{S}\left(\frac{\partial u}{\partial x}\right)^2},\quad H_v=\sqrt{\mathcal{S}\left(\frac{\partial v}{\partial x}\right)^2},\quad H_w=\sqrt{\mathcal{S}\left(\frac{\partial w}{\partial x}\right)^2}. \tag{4.99}$$

If the system of coordinates is orthogonal its Lamé coefficients are reciprocal in magnitude to the corresponding differential parameters of the first order:

$$L_u=\frac{1}{H_u},\quad L_v=\frac{1}{H_v},\quad L_w=\frac{1}{H_w}. \tag{4.100}$$

The elements ds of length of the space curve, $d\sigma$ of the area of surface and $d\omega$ of volume of a body in the system of orthogonal curvilinear coordinates are expressed as follows:

$$ds = \sqrt{L_u^2\,du^2 + L_v^2\,dv^2 + L_w^2\,dw^2}$$
$$= \sqrt{\frac{1}{H_u^2}\,du^2 + \frac{1}{H_v^2}\,dv^2 + \frac{1}{H_w^2}\,dw^2}, \tag{4.101}$$

$$d\sigma = \sqrt{(L_uL_v\,du\,dv)^2 + (L_uL_w\,du\,dw)^2 + (L_vL_w\,dv\,dw)^2}, \tag{4.102}$$

$$d\omega = L_uL_vL_w\,du\,dv\,dw. \tag{4.103}$$

In particular, for elements ds_u, ds_v and ds_w – of the lengths of coordinate lines – we obtain the formulae

$$ds_u = L_u du, \quad ds_v = L_v dv, \quad ds_w = L_w dw. \tag{4.104}$$

If the surface S in the space $Oxyz$ is defined by the parametric equations

$$x = \varphi(u, v), \quad y = \psi(u, v), \quad z = \chi(u, v), \tag{4.105}$$

the element $d\sigma$ of its area can be expressed by the formula

$$d\sigma = \sqrt{EG - F^2}\,du\,dv, \tag{4.106}$$

where

$$E = S\left(\frac{\partial x}{\partial u}\right)^2, \quad F = S\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right), \quad G = S\left(\frac{\partial x}{\partial v}\right)^2. \tag{4.107}$$

The magnitudes E, F and G are called the *Gaussian coefficients* of the surface S in the given system of curvilinear coordinates u, v. The element ds of length of line on the surface S equals

$$ds = \sqrt{S\left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right)^2} = \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}. \tag{4.108}$$

Certain Systems of Curvilinear Coordinates in Space

2. The description of common systems of curvilinear coordinates in space is given below, according to the following scheme: for orthogonal systems, we are given (a) their definition (i.e. the expression of rectangular cartesian coordinates x, y, z, through curvi-

linear coordinates u, v, and w, (b) coordinate surfaces, (c) formulae for Lamé coefficients L_u, L_v, L_w (the differential parameters of the first order H_u, H_v, H_w for orthogonal systems are calculated as their reciprocals), (d) the formula for the element ds of the length of a line, (e) the formula for the element $d\sigma$ of the area of surface, (f) the formula for the element $d\omega$ of volume. For non-orthogonal systems, the description of their principal properties is given. In some necessary cases certain explanations are included in the text.

1°. Cartesian Coordinates

(a)

$$\left.\begin{aligned} x &= a_1'u + b_1'v + c_1'w + d_1', \\ y &= a_2'u + b_2'v + c_2'w + d_2', \\ z &= a_3'u + b_3'v + c_3'w + d_3', \end{aligned}\right\} \tag{4.109}$$

where a', b', c', d' are constant and the Jacobian $J_1 = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is a non-zero one. This system is transformed into the following one:

$$\left.\begin{aligned} u &= a_1x + b_1y + c_1z + d_1, \\ v &= a_2x + b_2y + c_2z + d_2, \\ w &= a_3x + b_3y + c_3z + d_3, \end{aligned}\right\} \tag{4.110}$$

where a, b, c, d are constants and the Jacobian $J = \dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 1/g_1$ is also non-zero.

A system of cartesian coordinates in general is not orthogonal, since the magnitudes

$$\left.\begin{aligned} S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) &= a_1a_2 + b_1b_2 + c_1c_2, \\ S\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right) &= a_1a_3 + b_1b_3 + c_1c_3, \\ S\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\right) &= a_2a_3 + b_2b_3 + c_2c_3 \end{aligned}\right\} \tag{4.111}$$

do not equal zero for all a, b, c.

The condition for the invariability of scales of an orthogonal system is that the equations

$$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2 = a_3^2 + b_3^2 + c_3^2$$
$$= (a_1'^2 + a_1'^2 + a_1'^2 = b_2'^2 + b_2'^2 + b_2'^2 = c_3'^2 + c_3'^2 + c_3'^2) = 1 \tag{4.112}$$

be satisfied.

(b) The coordinate surfaces are *planes* parallel to the planes

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0. \end{aligned}\right\} \tag{4.113}$$

$$\left.\begin{aligned} \text{(c)}\quad L_u &= \sqrt{a_1'^2 + a_2'^2 + a_3'^2} = \frac{1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \\ L_v &= \sqrt{b_1'^2 + b_2'^2 + b_3'^2} = \frac{1}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \\ L_w &= \sqrt{c_1'^2 + c_2'^2 + c_3'^2} = \frac{1}{\sqrt{a_3^2 + b_3^2 + c_3^2}}. \end{aligned}\right\} \tag{4.114}$$

(Here, and below, it is assumed that the system is orthogonal.)

$$\begin{aligned} \text{(d)}\quad ds^2 &= (a_1'^2 + a_2'^2 + a_3'^2)\,du^2 + (b_1'^2 + b_2'^2 + b_3'^2)\,dv^2 \\ &\quad + (c_1'^2 + c_2'^2 + c_3'^2)\,dw^2 = \frac{1}{a_1^2 + b_1^2 + c_1^2}\,du^2 \\ &\quad + \frac{1}{a_2^2 + b_2^2 + c_2^2}\,dv^2 + \frac{1}{a_3^2 + b_3^2 + c_3^2}\,dw^2. \end{aligned} \tag{4.115}$$

$$\begin{aligned} \text{(e)}\quad d\sigma^2 &= (a_1'^2 + a_2'^2 + a_3'^2)(b_1'^2 + b_2'^2 + b_3'^2)(du\,dv)^2 \\ &\quad + (a_1'^2 + a_2'^2 + a_3'^2)(c_1'^2 + c_2'^2 + c_3'^2)(du\,dw)^2 \\ &\quad + (b_1'^2 + b_2'^2 + b_3'^2)(c_1'^2 + c_2'^2 + c_3'^2)(dv\,dw)^2 \\ &= \frac{(du\,dv)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)} \\ &\quad + \frac{(du\,dw)^2}{(a_1^2 + b_1^2 + c_1^2)(a_3^2 + b_3^2 + c_3^2)} \\ &\quad + \frac{(dv\,dw)^2}{(a_2^2 + b_2^2 + c_2^2)(a_3^2 + b_3^2 + c_3^2)}. \end{aligned} \tag{4.116}$$

(f) $$d\omega^2 = (a_1'^2 + a_2'^2 + a_3'^2)(b_1'^2 + b_2'^2 + b_3'^2) \times \\ \times (c_1'^2 + c_2'^2 + c_3'^2)\, du^2\, dv^2\, dw^2 \\ = \frac{du^2\, dv^2\, dw^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)(a_3^2 + b_3^2 + c_3^2)}. \tag{4.117}$$

2°. Cylindrical Coordinates

(a) $$\left.\begin{aligned} x &= u\cos v, \\ y &= u\sin v, \\ z &= w, \end{aligned}\right\} \tag{4.118}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad -\infty < w < +\infty.$$

The system is orthogonal.

(b) The coordinate surfaces are *circular cylinders*

$$x^2 + y^2 = u_0^2, \tag{4.119}$$

whose axes coincide with the axis Oz and the radius equals u_0, the *half-planes*

$$x = u\cos v_0, \quad y = u\sin v_0, \tag{4.120}$$

which pass through the axis Oz and forming an angle v_0 with the plane Oxz, and the *planes*

$$z = w_0, \tag{4.121}$$

parallel to the plane Oxy.

(c) $$L_u = 1, \quad L_v = u, \quad L_w = 1. \tag{4.122}$$

(d) $$ds = \sqrt{du^2 + u^2 dv^2 + dw^2}. \tag{4.123}$$

(e) $$d\sigma = \sqrt{u^2(du\, dv)^2 + (du\, dw)^2 + u^2(dv\, dw)^2}. \tag{4.124}$$

(f) $$d\omega = u\, du\, dv\, dw. \tag{4.125}$$

Often, generalized cylindrical coordinates u, v, w of point $p(x, y, z)$ are used, connected with x, y, z as follows:

$$\left.\begin{aligned} x &= au\cos v, \\ y &= bu\sin v, \\ z &= cw, \end{aligned}\right\} \tag{4.126}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad -\infty < w < +\infty,$$

where a, b, c are positive constants. When $a \neq b$ the system is non-orthogonal. For this system

$$d\omega = \left| \frac{\partial(x, y, z)}{\partial(u, v, \omega)} \right| du\, dv\, dw = abcu\, du\, dv\, dw. \qquad (4.127)$$

In general, the name orthogonal *cylindrical* can be given to any system of curvilinear coordinates of the form

$$x = \varphi(u, v), \quad y = \psi(u, v), \quad z = w, \qquad (4.128)$$

where the first two relationships define some orthogonal system of coordinates in the plane Oxy. For such space systems, the coordinate surfaces are *planes* parallel to the plane Oxy (w = const) and *cylindrical* surfaces, whose generators are parallel to the axis Oz.

3°. Spherical Coordinates

(a)

$$\left.\begin{aligned} x &= u \cos v \sin w, \\ y &= u \sin v \sin w, \\ z &= u \cos w, \end{aligned}\right\} \qquad (4.129)$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad 0 \leqslant w \leqslant \pi.$$

The system is orthogonal.

(b) The coordinate surfaces are *spheres*

$$x^2 + y^2 + z^2 = u_0^2 \qquad (4.130)$$

with centre at the origin and of radius u_0, *half-planes*

$$x = u \cos v_0 \sin w, \quad y = u \sin v_0 \sin w, \qquad (4.131)$$

which pass through the axis Oz at an angle v_0 to the plane Oxz, and *circular cones*

$$x^2 + y^2 = z^2 \tan^2 w_0 \qquad (4.132)$$

($z = 0$ when $w_0 = \pi/2$) with vertex at the origin and generator inclined to the axis Oz at an angle w_0.

(c)

$$L_u = 1, \quad L_v = u \sin w, \quad L_w = u. \qquad (4.133)$$

(d)

$$ds = \sqrt{du^2 + u^2 \sin^2 w\, dv^2 + u^2\, dw^2}. \qquad (4.134)$$

(e) $d\sigma = \sqrt{u^2 \sin^2 w(du\,dv)^2 + u^2(du\,dw)^2 + u^4 \sin^2 w(dv\,dw)^2}$. (4.135)

(f) $d\omega = u^2 \sin w\, du\, dv\, dw$. (4.136)

Use is often made of *generalized* spherical coordinates u, v, w of the point (x, y, z) connected with x, y, z, as follows

$$\left.\begin{aligned} x &= au \cos v \sin w, \\ y &= bu \sin v \sin w, \\ z &= cu \cos w, \end{aligned}\right\} \qquad (4.137)$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad 0 \leqslant w \leqslant \pi,$$

where a, b, c are positive constants. For $a \neq b$ the system is not orthogonal. For it

$$d\omega = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw = abcu^2 \sin w\, du\, dv\, dw. \qquad (4.138)$$

4⁰. General Ellipsoidal Coordinates

(a)
$$\left.\begin{aligned} x^2 &= \frac{(u + a^2)(v + a^2)(w + a^2)}{(b^2 - a^2)(c^2 - a^2)}, \\ y^2 &= \frac{(u + b)^2 (v + b^2)(w + b^2)}{(a^2 - b^2)(c^2 - b^2)}, \\ z^2 &= \frac{(u + c^2)(v + c^2)(w + c^2)}{(a^2 - c^2)(b^2 - c^2)}, \end{aligned}\right\} \qquad (4.139)$$

$$0 \leqslant c^2 < b^2 < a^2, \quad -a^2 < w < -b^2 < v < -c^2 < u < +\infty.$$

The system is orthogonal.

(b) The coordinate surfaces are confocal *ellipsoids*

$$\frac{x^2}{u_0 + a^2} + \frac{y^2}{u_0 + b^2} + \frac{z^2}{u_0 + c^2} = 1, \qquad (4.140)$$

hyperboloids of one sheet

$$\frac{x^2}{v_0 + a^2} + \frac{y^2}{v_0 + b^2} - \frac{z^2}{-(v_0 + c^2)} = 1, \qquad (4.141)$$

hyperboloids of two sheets

$$\frac{x^2}{w_0 + a^2} - \frac{y^2}{-(w_0 + b^2)} - \frac{z^2}{-(w_0 + c^2)} = 1. \quad (4.142)$$

$$\left.\begin{aligned} \text{(c) } L_u^2 &= \frac{1}{4}\frac{(u-v)(u-w)}{M(u)}, \quad L_v^2 = \frac{1}{4}\frac{(u-v)(v-w)}{-M(v)}, \\ L_w^2 &= \frac{1}{4}\frac{(u-w)(v-w)}{M(w)}, \end{aligned}\right\} \quad (4.143)$$

where $M(t) = (t + a^2)(t + b^2)(t + c^2)$, so that $M(u) > 0$, $M(v) < 0$, $M(w) > 0$.

$$\text{(d)} \quad ds = \frac{1}{2} \times$$

$$\times \sqrt{\frac{(u-v)(u-w)}{M(u)}du^2 + \frac{(u-v)(v-w)}{-M(v)}dv^2 + \frac{(u-w)(v-w)}{M(w)}dw^2}. \quad (4.144)$$

$$\text{(e)} \quad d\sigma = \frac{1}{4}\left[\frac{(u-v)^2(u-w)(v-w)}{-M(u)\,M(v)}(du\,dv)^2 + \frac{(u-v)(u-w)^2(v-w)}{M(u)\,M(w)}(du\,dw)^2 + \frac{(u-v)(u-w)(v-w)^2}{-M(v)\,M(w)}(dv\,dw)^2\right]^{\frac{1}{2}}. \quad (4.145)$$

$$\text{(f)} \quad d\omega = \frac{1}{8}\frac{(u-v)(u-w)(v-w)}{\sqrt{-M(u)\,M(v)\,M(w)}}du\,dv\,dw. \quad (4.146)$$

Ellipsoidal coordinates depend on the parameters a^2, b^2, c^2, which obey the conditions $a^2 > b^2 > c^2 \geqslant 0$. Extending the definition of coordinates to limit cases, i.e. to cases in which one, two or three of these inequalities are changed to equations, we arrive at systems of coordinates which can be usefully regarded as degenerate cases of the system of *general ellipsoidal coordinates*. These degenerate

systems remain orthogonal. In particular, spherical coordinates (see § 3, sec. 2, 3⁰) are degenerate ellipsoidal coordinates in the above sense: the first are obtained from the latter, if we take $c = 0$, put $\bar{u} = u^2$, $\bar{v} = b^2 \sin^2 v$, $w = -(a^2 - b^2)\sin^2 w - b^2$, where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are general ellipsoidal coordinates, and then we make first b and then a tend to zero.

Other cases of degenerate ellipsoidal coordinates are given in the following two sections.

5^0. Degenerate Ellipsoidal "Elongated" Coordinates

(a)
$$\left.\begin{aligned} x &= \sinh u \cos v \sin w, \\ y &= \sinh u \sin v \sin w, \\ z &= \cosh u \cos w, \end{aligned}\right\} \tag{4.147}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad 0 \leqslant w \leqslant \pi.$$

The system is orthogonal. It is obtained from the general ellipsoidal coordinates $\bar{u}, \bar{v}, \bar{w}$ if we take $c = 0$, put

$$\bar{u} = a^2 \sinh^2 u, \quad \bar{v} = -b^2 \sin^2 v, \quad \bar{w} = -(a^2 - b^2)\sin^2 w - b^2 \tag{4.148}$$

and regard b as tending to zero and $a = 1$. (After that, it is still necessary to interchange the notation of coordinates x, y and z.)

(b) The coordinate surfaces are "elongated" *ellipsoids of rotation*

$$\frac{x^2 + y^2}{\sinh^2 u_0} + \frac{z^2}{\cosh^2 u_0} = 1, \quad u_0 \neq 0, \tag{4.149}$$

half-planes

$$y = (\tan v_0)x, \quad v_0 \neq \frac{\pi}{2}, \quad \frac{3\pi}{2}, \tag{4.150}$$

and *two-sheeted hyperboloids of revolution*

$$-\frac{x^2 + y^2}{\sin^2 w_0} + \frac{z^2}{\cos^2 w_0} = 1, \quad w_0 \neq 0, \quad \frac{\pi}{2}, \quad \pi. \tag{4.151}$$

(c)
$$\left.\begin{aligned} L_u &= \sqrt{\sinh^2 u + \sin^2 w}, \quad L_v = \sinh u \sin w, \\ L_w &= \sqrt{\sinh^2 u + \sin^2 w}. \end{aligned}\right\} \tag{4.152}$$

(d) $ds = \sqrt{(\sinh^2 u + \sin^2 w)(du^2 + dw^2) + \sinh^2 u \sin^2 w\, dv^2}.$ (4.153)

(e) $d\sigma$

$$= \sqrt{(\sinh^2 u + \sin^2 w)\sinh^2 u \sin^2 w (du^2 + dw^2)\, dv^2 + (\sinh^2 u + \sin^2 w)^2 (du\, dw)^2}. \tag{4.154}$$

(f) $d\omega = (\sinh^2 u + \sin^2 w) \sinh u \sin w\, du\, dv\, dw.$ (4.155)

6⁰. Degenerate Ellipsoidal "Flattened" Coordinates

(a)

$$\left.\begin{aligned} x &= \cosh u \cos v \sin w, \\ y &= \cosh u \sin v \sin w, \\ z &= \sinh u \cos w, \end{aligned}\right\} \tag{4.156}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant \sinh v < 2\pi, \quad 0 \leqslant w \leqslant \pi.$$

The system is orthogonal. It is obtained from the system of general ellipsoidal coordinates $\bar{u}, \bar{v}, \bar{w}$, if we take $c = 0$, put

$$\bar{u} = a^2 \sinh^2 u, \quad \bar{v} = -b^2 \cos^2 v, \quad \bar{w} = -(a^2 - b^2)\cos^2 w - b^2 \tag{4.157}$$

and regard $b = 1$ and $a \to 1$.

(b) The coordinate surfaces are "flattened" *ellipsoids of rotation*

$$\frac{x^2 + y^2}{\cosh^2 u_0} + \frac{z^2}{\sinh^2 u_0} = 1, \quad u_0 \neq 0, \tag{4.158}$$

the *half-planes*

$$y = (\tan v_0)x, \quad v_0 \neq \frac{\pi}{2}, \ \frac{3\pi}{2}, \tag{4.159}$$

and *one-sheeted hyperboloids of revolution*

$$\frac{x^2 + y^2}{\sin^2 w_0} - \frac{z^2}{\cos^2 w_0} = 1, \quad w_0 \neq 0, \ \frac{\pi}{2}, \pi. \tag{4.160}$$

(c)

$$\left.\begin{aligned} L_u = \sqrt{\sinh^2 u + \cos^2 w}, \quad L_v &= \cosh u \sin w, \\ L_w = \sqrt{\sinh^2 u + \cos^2 w}. \end{aligned}\right\} \tag{4.161}$$

(d) $ds = \sqrt{(\sinh^2 u + \cos^2 w)(du^2 + dw^2) + \sinh^2 u \sin^2 w \, dv^2}.$ (4.162)

(e) $d\sigma = [(\sinh^2 u + \cos^2 w)\cosh^2 u \sin^2 w(du^2 + dw^2)\, dv^2$

$+ (\sinh^2 u + \cos^2 w)^2 (du\, dw)^2]^{1/2}.$ (4.163)

(f) $d\omega = (\sinh^2 u + \cos^2 w)\cosh u \sin w \, du\, dv\, dw.$ (4.164)

7°. Sphero-Conical Coordinates

(a)
$$\left.\begin{aligned} x^2 &= \frac{u(v + a^2)(w + a^2)}{(a^2 - b^2)a^2}, \\ y^2 &= \frac{u(v + b^2)(w + b^2)}{(b^2 - a^2)b^2}, \\ z^2 &= \frac{uvw}{a^2b^2}, \end{aligned}\right\} \tag{4.165}$$

$$-a^2 < w < -b^2 < v < 0 < u < +\infty.$$

The system is orthogonal.

(b) The coordinate surfaces are *spheres*

$$x^2 + y^2 + z^2 = u_0 \tag{4.166}$$

with centre at the origin, *elliptical cones*

$$\frac{x^2}{v_0 + a^2} + \frac{y^2}{v_0 + b^2} - \frac{z^2}{-v_0} = 0 \tag{4.167}$$

with vertex at the origin and axis situated on the axis Oz, and *elliptical cones*

$$\frac{x^2}{w_0 + a^2} - \frac{y^2}{-(w_0 + b^2)} - \frac{z^2}{-w_0} = 0 \tag{4.168}$$

with vertex at the origin and axis situated on the axis Ox.

(c)
$$\left.\begin{aligned} &L_u = \frac{1}{2\sqrt{u}}, \quad L_v = \frac{1}{2}\sqrt{\frac{u(v - w)}{-N(v)}}, \\ &L_w = \frac{1}{2}\sqrt{\frac{u(v - w)}{N(w)}}, \end{aligned}\right\} \tag{4.169}$$

where $N(t) = t(t + a^2)(t + b^2)$, so that $N(v) < 0$, $N(w) > 0$.

(d) $$ds = \frac{1}{2}\sqrt{\frac{du^2}{u} + \frac{u(v-w)}{-N(v)}\,dv^2 + \frac{u(v-w)}{N(w)}\,dw^2}. \quad (4.170)$$

(e)
$$d\sigma = \frac{1}{4}\sqrt{\frac{v-w}{-N(v)}(du\,dv)^2 + \frac{v-w}{N(w)}(du\,dw)^2 + \frac{u^2(v-w)^2}{-N(v)N(w)}(dv\,dw)^2}. \quad (4.171)$$

(f) $$d\omega = \frac{u(v-w)}{8\sqrt{-uN(v)N(w)}}\,du\,dv\,dw. \quad (4.172)$$

8^0. Parabolical Coordinates

(a) $$\left.\begin{aligned} x &= 2uw\cos v, \\ y &= 2uw\sin v, \\ z &= u^2 - w^2, \end{aligned}\right\} \quad (4.173)$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad 0 \leqslant w < +\infty.$$

The system is orthogonal.

(b) The coordinate surfaces are *paraboloids of revolution*

$$\frac{x^2 + y^2}{4u_0^2} = u_0^2 - z, \quad u_0 \neq 0, \quad (4.174)$$

with vertex at point $(0, 0, u_0^2)$ obtained on rotating parabolae $y^2/(4u_0^2) = u_0^2 - z$ about the ray $(-\infty, u_0^2)$ of the axis Oz, the *half-planes*

$$y = (\tan v_0)x, \quad v_0 \neq \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad (4.175)$$

passing through the axis Oz and forming an angle v_0 with the positive $(x > 0)$ half-plane Oxz, and *paraboloids of revolution*

$$\frac{x^2 + y^2}{4w_0^2} = w_0^2 + z, \quad w_0 \neq 0, \quad (4.176)$$

with vertex at point $(0, 0, -w_0^2)$ obtained on rotating parabolae $y^2/(4w_0^2) = w_0^2 + z$ about the ray $(-w_0^2, +\infty)$ of the axis Oz.

(c) $$L_u = 2\sqrt{u^2 + w^2}, \quad L_v = 2uw, \quad L_w = 2\sqrt{u^2 + w^2}. \tag{4.177}$$

(d) $$ds = 2\sqrt{(u^2 + w^2)(du^2 + dw^2) + u^2w^2\,dv^2}. \tag{4.178}$$

(e) $$d\sigma = 4\sqrt{(u^2 + w^2)u^2w^2(du^2 + dw^2)dv^2 + (u^2 + w^2)^2\,(du\,dw)^2}. \tag{4.179}$$

(f) $$d\omega = 8(u^2 + w^2)uw\,du\,dv\,dw. \tag{4.180}$$

9°. Toroidal Coordinates

$$\left.\begin{aligned} x &= \frac{\sinh u \cos v}{\cosh u - \cos w}, \\ y &= \frac{\sinh u \sin v}{\cosh u - \cos w}, \\ z &= \frac{\sin w}{\cosh u - \cos w}, \end{aligned}\right\} \tag{4.181}$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v < 2\pi, \quad -\pi \leqslant w \leqslant \pi.$$

The system is orthogonal.

(b) The coordinate surfaces are *toruses*

$$\left(\sqrt{x^2 + y^2} - \coth u_0\right)^2 + z^2 = \frac{1}{\sinh^2 u_0}, \quad u_0 \neq 0, \tag{4.182}$$

obtained by rotating the circle $(y - \coth u_0)^2 + z^2 = 1/\sinh^2 u_0$ about the axis Oz (this circle does not intersect the axis Oz, since $\coth u_0 > 1/\sinh u_0$), the *half-planes*

$$y = (\tan v_0)x, \quad v_0 \neq \frac{\pi}{2}, \ \frac{3\pi}{2}, \tag{4.183}$$

passing through the axis Oz and forming an angle v_0 with the positive $(x > 0)$ half-plane Oxz, and *spheres*

$$x^2 + y^2 + (z - \cot w_0)^2 = \frac{1}{\sin^2 w_0}, \quad w_0 \neq 0, \tag{4.184}$$

with centre at point $(0, 0, \cot w_0)^2$ and radius equal $1/|\sin w_0|$.

(c) $$L_u = \frac{1}{\cosh u - \cos w}, \quad L_v = \frac{\sinh u}{\cosh u - \cos w},$$

$$L_w = \frac{1}{\cosh u - \cos w}. \tag{4.185}$$

(d) $$ds = \frac{1}{\cosh u - \cos w}\sqrt{du^2 + \sinh^2 u\, dv^2 + dw^2}. \tag{4.186}$$

(e)

$$d\sigma = \frac{1}{(\cosh u - \cos w)^2}\sqrt{\sinh^2 u(du^2 + dw^2)\, dv^2 + (du\, dw)^2}. \tag{4.187}$$

(f) $$d\omega = \frac{\sinh u}{(\cosh u - \cos w)^3}\, du\, dv\, dw. \tag{4.188}$$

10°. Bipolar Coordinates

(a)
$$\left.\begin{aligned} x &= \frac{\sin u \cos v}{\cosh w - \cos u}, \\ y &= \frac{\sin u \sin v}{\cosh w - \cos u}, \\ z &= \frac{\sinh w}{\cosh w - \cos u}, \end{aligned}\right\} \tag{4.189}$$

$$0 \leqslant u < \pi, \quad 0 \leqslant v < 2\pi, \quad -\infty < w < +\infty.$$

The system is orthogonal.

(b) The coordinate surfaces are the *surfaces of revolution*

$$\left(\sqrt{x^2 + y^2} - \cot u_0\right)^2 + z^2 = \frac{1}{\sin^2 u_0}, \quad u_0 \neq 0, \tag{4.190}$$

obtained by rotating the circle $(y - \cot u_0)^2 + z^2 = 1/\sin^2 u_0$ about the axis Oz (this circle does intersect the axis Oz, since $|\cot u_0| < 1/\sin u_0$), *the half-planes*

$$y = (\tan v_0)x, \quad v_0 \neq \frac{\pi}{2}, \quad \frac{3\pi}{2}, \tag{4.191}$$

passing through the axis Oz and forming the angle v_0 with the positive ($x > 0$) half-planes Oxz, and *spheres*

$$x^2 + y^2 + (z - \coth w_0)^2 = \frac{1}{\sinh^2 w_0}, \quad w_0 \neq 0, \tag{4.192}$$

with centre at point $(0, 0, \coth w_0)$ and radius equal to $1/|\sinh w_0|$.

(c)
$$L_u = \frac{1}{\cosh w - \cos u}, \quad L_v = \frac{\sin u}{\cosh w - \cos u},$$

$$L_w = \frac{1}{\cosh w - \cos u}. \tag{4.193}$$

(d)
$$ds = \frac{1}{\cosh w - \cos u} \sqrt{du^2 + \sin^2 u \, dv^2 + dw^2}. \tag{4.194}$$

(e)

$$d\sigma = \frac{1}{(\cosh w - \cos u)^2} \sqrt{\sin^2 u(du^2 + dw^2)dv^2 + (du\, dw)^2}. \tag{4.195}$$

(f)
$$d\omega = \frac{\sin u}{(\cosh w - \cos u)^3} \, du \, dv \, dw. \tag{4.196}$$

CHAPTER V

THE INTEGRATION OF FUNCTIONS

§ 1. The Indefinite Integral

1. The primitive (otherwise, the *original* or *anti-derivative*) of the function $f(x)$ defined in the interval $[a, b]$ is the name given to a function $F(x)$ defined in the same interval and satisfying the condition:

$$F'(x) = f(x) \quad \text{or} \quad dF(x) = f(x)\,dx.$$

THEOREM 1. *If the function $f(x)$ is continuous at every point of the interval $[a, b]$, then its primitive exists and is continuous at every point of this interval. If the function $f(x)$ has discontinuities at certain points of the interval $[a, b]$, then its primitive exists and is continuous everywhere in $[a, b]$, except, perhaps, at points of discontinuity of $f(x)$.*

At points of removable discontinuity or of discontinuity of the first kind (a finite jump) the primitive is continuous. When there is an infinite discontinuity in $f(x)$, the primitive $F(x)$ may be continuous or may have a discontinuity.

A given function $f(x)$ has an infinite number of primitives, which differ from each other in their constant term only; i.e. *the difference between two primitives of one function is a constant.* The graphs of all primitives of a given function can be obtained from one of them by means of a parallel shift along the axis Oy.

The general expression $F(x) + C$, where C is an arbitrary constant for all primitives of the given function $f(x)$ is called its *indefinite integral*

$$F(x) + C = \int f(x)\,dx.$$

2. The indefinite integrals of the principal elementary functions can be obtained by the inversion of the formulae for derivatives. In more complex cases, the integral of a given function can be trans-

formed into the integral of other functions with the help of the following *properties of the indefinite integral:*

(a) $$\int kf(x)\,dx = k\int f(x)\,dx \quad (k = \text{const},\ k \neq 0),$$

i.e. a constant factor can be taken outside the integral sign.

(b) $$\int [f(x) + \varphi(x) - \psi(x)]\,dx = \int f(x)\,dx + \int \varphi(x)\,dx - \int \psi(x)\,dx,$$

i.e. the integral of an algebraic sum of a finite number of functions equals the sum of the integrals of the separate terms.

(c) $$\int f(x)\,dx = \int f[\varphi(t)]\,\varphi'(t)\,dt,$$

where $x = \varphi(t)$; this equation is called *the rule of substitution* or *change of variable.*

(d) $$\int f(x)\,\varphi'(x)\,dx = f(x)\,\varphi(x) - \int \varphi(x) f'(x)\,dx$$

is *the formula of integration by parts.* It should be kept in mind that the equations (a), (b), (c), (d) mean that both sides of each equation are sets of primitives of the same function. In order to verify such an equation it is sufficient to check that the derivatives of the left sides are equal to those of the right sides.

3. The formula of integration by parts can be generalized to a form with whose help the finding of the integral of the product $f(x)\,\varphi^n(x)$ is reduced to the integration of the product $f^{(n)}(x)\,\varphi(x)$. Such a formula is called *the formula of successive integration by parts* and has the form

$$\int f(x)\,\varphi^{(n)}(x)\,dx = f(x)\,\varphi^{(n-1)}(x) - f'(x)\,\varphi^{(n-2)}(x) + f''(x)\,\varphi^{(n-3)}(x) - \cdots + (-1)^{n-1} f^{(n-1)}(x)\,\varphi(x) + (-1)^n \int f^{(n)}(x)\,\varphi(x)\,dx.$$

As an example of the application of this formula let us work out the integral

$$\int e^{ax} P_n(x)\,dx,$$

where $P_n(x)$ is a polynomial of degree n. Putting $\varphi(x) = e^{ax}/a^n$ we note that $\varphi^{(n)}(x) = e^{ax}$, whence, on applying the formula for successive integration by parts, we find that

$$\int e^{ax} P_n(x)\,dx = e^{ax}\left\{\frac{P_n(x)}{a} - \frac{P_n'(x)}{a^2} + \frac{P_n''(x)}{a^3} + \cdots + (-1)^n \frac{P_n^{(n)}(x)}{a^{n+1}}\right\} + C.$$

It is often necessary to find several *successive primitives* for the function $f(x)$; this necessity arises, for example, when a differential equation of the form $y^{(n)} = f(x)$ is integrated.

The expression for the nth successive primitive is given by the *Cauchy formula*

$$\int dx \int dx \ldots \int f(x)\, dx = \frac{1}{(n-1)!} \int (x-t)^{n-1} f(t)\, dt + P_{n-1}(x),$$

where after integrating, t is changed to x on the right, and $P_{n-1}(x)$ is a polynomial of degree $n - 1$ with arbitrary coefficients.

§ 2. The Integration of Elementary Functions

1. As is already known (see Chapter I) the operation of differentiation of elementary functions always leads to elementary functions. In carrying out integration—an operation which is inverse to that of differentiation—the situation is entirely different. The integration of rational functions always leads to a sum of a finite number of elementary functions, but the integrals of other elementary functions often cannot be expressed by a finite number of elementary functions. Such integrals serve as one of the sources out of which arise new non-elementary functions, so-called *higher transcendental functions.* These are often encountered in practice and are thoroughly well known and tabulated; they are used in mathematics and its applications just as much as the elementary ones are.

2. The integral of a rational function is either rational or is represented by a sum of a rational function and a finite number of logarithms and arctangents of rational functions with constant coefficients in front of the sign of logarithm and arctangent.

1°. An entire rational function of the nth degree is integrated directly and leads to another entire rational function of degree $n + 1$:

$$\begin{aligned}\int (a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n)\, dx \\ = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \frac{a_2}{n-1} x^{n-1} + \cdots \\ + \frac{a_{n-1}}{2} x^2 + a_n x + C.\end{aligned} \tag{5.1}$$

2^0. The integral of the simplest fraction with a linear denominator is a logarithmic function

$$\int \frac{dx}{x - x_0} = \ln |x - x_0| + C, \tag{5.2}$$

and that of the simplest fraction with a quadratic denominator, which has complex roots, is the sum of a logarithmic function and arctangent:

$$\int \frac{Mx + N}{x^2 + px + q} dx$$

$$= \frac{M}{2} \ln |x^2 + px + q| + \frac{N - \frac{Mp}{2}}{\sqrt{q - \frac{p^2}{4}}} \arctan \frac{x + \frac{p}{2}}{\sqrt{q - \frac{p^2}{4}}} + C$$

$$\left(q - \frac{p^2}{4} > 0\right). \tag{5.3}$$

3^0. If the denominator is a linear binomial of degree k, the integral is a fractional rational function

$$\int \frac{dx}{(x - x_0)^k} = - \frac{1}{(k - 1)(x - x_0)^{k-1}} + C \quad (k \neq 1). \tag{5.4}$$

If the denominator has complex conjugate k-fold roots, it is possible to apply the recurrence formula, having first transformed the fraction under the integral sign thus:

$$\frac{M_k x + N_k}{(x^2 + px + q)^k} = \frac{M_k}{2} \frac{(2x + p)}{(x^2 + px + q)^k} + \left(N_k - \frac{M_k p}{2}\right) \frac{1}{(x^2 + px + q)^k}$$

and having split up the integral into a sum of two terms. The first term is integrated directly and leads to a fractional rational function

$$\frac{M_k}{2} \int \frac{2x + p}{(x^2 + px + q)^k} dx = - \frac{M_k}{2(k - 1)} \frac{1}{(x^2 + px + q)^{k-1}} + C, \tag{5.5}$$

and the integral of the second term leads to the sum of a fractional rational function and an analogous integral of a fraction with a denominator $(x^2 + px + q)^{k-1}$:

$$\int \frac{dx}{(x^2 + px + q)^k} = \frac{x + \frac{p}{2}}{2(k-1)\left(q - \frac{p^2}{4}\right)(x^2 + px + q)^{k-1}}$$

$$+ \frac{2k-3}{2(k-1)\left(q - \frac{p^2}{4}\right)} \int \frac{dx}{dx\,(x^2 + px + q)^{k-1}}. \tag{5.6}$$

A second application of this recurrence formula leads to the integral

$$\int \frac{dx}{x^2 + px + q} = \frac{1}{\sqrt{q - \frac{p^2}{4}}} \arctan \frac{x + \frac{p}{2}}{\sqrt{q - \frac{p^2}{4}}} + C$$

$$\left(q - \frac{p^2}{4} > 0\right), \tag{5.7}$$

and, therefore, the integral $\int \frac{M_k x + N_k}{(x^2 + px + q)^k}\,dx$ is also an elementary function.

4°. The integration of rational functions, in the general case, is carried out as follows. If the rational fraction $F(x)/Q(x)$ is irregular (i.e. the degree of the polynomial $F(x)$ is greater or equal to the degree of polynomial $Q(x)$) then on dividing $F(x)$ by $Q(x)$ according to the rule of dividing a polynomial by another polynomial, it is possible to represent the fraction $F(x)/Q(x)$ in which *the coefficient of the highest power of the denominator can always be made equal unity.*

A regular rational fraction $P(x)/Q(x)$ can be expanded into a sum of partial fractions of form $\frac{A_{ik}}{(x - x_i)^k}$ and $\frac{M_{ik}x + N_{ik}}{(x^2 + p_i x + q_i)^k}$ which can be integrated in the manner indicated above.

In particular, if all zeros $x_1, x_2, \ldots, x_n$ of a polynomial of nth degree $Q(x)$ are real and different, then

$$Q(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$$

and

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \cdots + \frac{A_n}{x - x_n}, \tag{5.8}$$

where coefficients A_i are determined from formulae

$$A_1 = \frac{P(x_1)}{Q'(x_1)}, \quad A_2 = \frac{P(x_2)}{Q'(x_2)}, \quad \ldots, \quad A_n = \frac{P(x_n)}{Q'(x_n)}. \tag{5.9}$$

In practice the coefficients A_{ik}, M_{ik}, N_{ik} are often found by the method of indeterminate coefficients.

In the general case, the fraction under the integral is represented in the form of a sum of partial fractions thus

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{(x - x_1)^2} + \cdots + \frac{A_l}{(x - x_1)^l} + \frac{B_1}{x - x_2}$$
$$+ \frac{B_2}{(x - x_2)^2} + \cdots + \frac{B_m}{(x - x_2)^m} + \cdots + \frac{M_1 x + N_1}{x^2 + px + q}$$
$$+ \frac{M_2 x + N_2}{(x^2 + px + q)^2} + \cdots + \frac{M_r x + N_r}{(x^2 + px + q)^r} + \cdots, \tag{5.10}$$

where

$$Q(x) = (x - x_1)^l (x - x_2)^m \ldots (x^2 + px + q)^r \ldots$$

EXAMPLE 1. Evaluate the integral $\int \frac{dx}{(x+1)^2 (x^2+1)^2}$.

We have

$$\frac{1}{(x+1)^2 (x^2+1)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2},$$

whence

$$1 \equiv A(x+1)(x^2+1)^2 + B(x^2+1)^2$$
$$+ (Cx+D)(x+1)^2(x^2+1) + (Ex+F)(x+1)^2.$$

We rewrite the latter identity in the form

$$1 \equiv (A+C)x^5 + (A+B+2C+D)x^4 + (2A+2C+2D+E)x^3$$
$$+ (2A+2B+2C+2D+2E+F)x^2 + (A+C+2D+E+2F)x$$
$$+ (A+B+D+F).$$

Comparing coefficients of equal powers of x on the left and on the right of the identity we obtain a set of equations

$$\begin{aligned} A + C &= 0, \\ A + B + 2C + D &= 0, \\ 2A + 2C + 2D + E &= 0, \\ 2A + 2B + 2C + 2D + 2E + F &= 0, \\ A + C + 2D + E + 2F &= 0, \\ A + B + D + F &= 1, \end{aligned}$$

whence we find

$$A = \tfrac{1}{2}, \quad C = -\tfrac{1}{2}, \quad B = D = \tfrac{1}{4}, \quad E = -\tfrac{1}{2}, \quad F = 0.$$

Therefore,

$$\int \frac{dx}{(x+1)^2(x^2+1)^2} = \frac{1}{2}\ln|x+1| - \frac{1}{4}\frac{1}{x+1} - \frac{1}{4}\ln(x^2+1) + \frac{1}{4}\arctan x + \frac{1}{4}\frac{1}{x^2+1} + C.$$

In integrating rational fractions it is also possible to apply the Ostrogradskii method, which enables us to reduce the integration of an arbitrary regular rational fraction to the integration of a fraction, whose denominator has simple roots only. For that, the denominator of the regular fraction $P(x)/Q(x)$ must be represented in the form

$$Q(x) = X_1(x)\, X_2^2(x) \ldots X_p^p(x), \tag{5.11}$$

where $X_1(x)$ is the product of linear and quadratic factors, corresponding to simple roots, and $X_i(x)$ is the product of linear and quadratic factors, corresponding to the i-fold roots. Then

$$\int \frac{P(x)}{Q(x)}\, dx \equiv \frac{H(x)}{U(x)} + \int \frac{G(x)}{V(x)}\, dx, \tag{5.12}$$

where

$$U(x) = X_2 X_3^2 \ldots X_p^{p-1}, \tag{5.13}$$

$$V(x) = X_1 X_2 \ldots X_p, \tag{5.14}$$

and $H(x)$ and $G(x)$ are polynomials with indeterminate coefficients of degrees less by one than those of the polynomials $U(x)$ and $V(x)$ respectively.

The polynomials $U(x)$ and $V(x)$ can be found, without expanding $Q(x)$ into its simplest factors: $U(x)$ is the greatest common divisor of the polynomials $Q(x)$ and $Q'(x)$ and $V(x) = Q'(x)/U(x)$.

The coefficients of the polynomials $H(x)$ and $G(x)$ can be found by differentiating the identity (5.12), which leads to a new identity

$$P \equiv \frac{(UH' - HU')V}{U} + GU. \tag{5.15}$$

The coefficients of equal powers of x on both sides of this identity should be compared and a set of algebraic equations should be obtained for finding the required coefficients of polynomials $H(x)$ and $G(x)$.

EXAMPLE 2. Calculate the integral

$$\int \frac{4x^5 - 1}{(x^5 + x + 1)^2}\, dx.$$

Here

$$Q(x) = (x^5 + x + 1)^2,$$

$$Q'(x) = 2(x^5 + x + 1)(5x^4 + 1),$$

$$U(x) = x^5 + x + 1,$$

$$V(x) = \frac{Q(x)}{U(x)} = x^5 + x + 1$$

and

$$\int \frac{4x^5 - 1}{(x^5 + x + 1)^2}\, dx = \frac{H(x)}{x^5 + x + 1} + \int \frac{G(x)}{x^5 + x + 1}\, dx.$$

Putting

$$H(x) = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4,$$

$$G(x) = B_0x^4 + B_1x^3 + B_2x^2 + B_3x + B_4,$$

we have

$$\begin{aligned} 4x^5 - 1 \equiv\ & (x^5 + x + 1)(B_0x^4 + B_1x^3 + B_2x^2 + B_3x + B_4) \\ & + (x^5 + x + 1)(4A_0x^3 + 3A_1x^2 + 2A_2x + A_3) \\ & - (5x^4 + 1)(A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4). \end{aligned}$$

Comparing the coefficients of equal powers of x on both sides of the latter identity, we find

$$A_0 = A_1 = A_2 = A_4 = 0, \quad A_3 = -1,$$

$$B_0 = B_1 = B_2 = B_3 = B_4 = 0,$$

and therefore

$$\int \frac{4x^5 - 1}{(x^5 + x + 1)^2}\, dx = -\frac{x}{x^5 + x + 1} + C.$$

3. The integral of an algebraic function, i.e. a function defined by the equation $f(x, y) = 0$, where $f(x, y)$ is a polynomial of x and y, may not be elementary.

In the case when the integral $\int R(x, y)\,dx$, where R is a rational function and y is an algebraic function of x, is elementary, it is either an algebraic function or a sum of an algebraic function and a finite number of logarithms and arctangents of algebraic functions with constant coefficients in front of the logarithm and arctangent signs, and all the algebraic functions of which the expression consists are rational with respect to x and y.

In order to clarify whether the integral $\int R(x, y)\,dx$ is an elementary function or not, it is necessary to investigate the curve, whose equation, $f(x, y) = 0$, defines the function y. If this curve is *rational*, i.e. the coordinates x and y of the points of this curve can be expressed as rational functions of some parameter t, then an appropriate substitution reduces $\int R(x, y)\,dx$ to an integral of a rational function, as a result of which the integral turns out to be an elementary function.

In the case when the curve $f(x, y) = 0$ is not rational, the integral $\int R(x, y)\,dx$ is not, generally speaking, an elementary function. It is necessary to keep in mind that the rationality of a curve is sufficient for the integral $\int R(x, y)\,dx$ to be elementary, but it is not necessary.

Some types of integrals of irrational functions can be reduced to integrals of rational functions by means of the above-mentioned substitutions.

1°. The integral

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_2}{n_2}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_p}{n_p}}\right] dx \tag{5.16}$$

or, in particular,

$$\int R\left[x, \ (ax+b)^{\frac{m_1}{n_1}}, \ (ax+b)^{\frac{m_2}{n_2}}, \ \ldots, \ (ax+b)^{\frac{m_p}{n_p}}\right] dx, \tag{5.17}$$

where R is a rational function of its arguments, and $m_1, m_2, \ldots, m_p$, $n_1, n_2, \ldots, n_p$ are integers, is reduced to an integral of a rational function by the substitution

$$\frac{ax+b}{cx+d} = t^N \tag{5.18}$$

(in the given particular case: $ax + bz + t^N$) where N is the least common multiple of numbers $n_1, n_2, \ldots, n_p$.

EXAMPLE 3. Calculate the integral $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x}$. Putting $\frac{1-x}{1+x} = t^2$ we have

$$x = \frac{1-t^2}{1+t^2}, \quad dx = -\frac{4t\,dt}{(1+t^2)^2}$$

and, therefore,

$$\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x} = \int \frac{-4t^2}{(1-t^2)(1+t^2)}\,dt.$$

On resolving the function under the integral sign into partial fractions, we get

$$\frac{-4t^2}{(1+t)(1-t)(1+t^2)} = -\frac{1}{1+t} - \frac{1}{1-t} + \frac{2}{1+t^2}$$

and further

$$\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x} = 2 \operatorname{arctg} t - \ln|1+t| + \ln|1-t| + C$$

$$= 2 \operatorname{arctg} \sqrt{\frac{1-x}{1+x}} + \ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right| + C.$$

EXAMPLE 4. Calculate the integral

$$\int \frac{dx}{\sqrt{1+x}+\sqrt[3]{1+x}}.$$

Putting $1 + x = t^6$, we have $dx = 6t^5\,dt$ and

$$\int \frac{dx}{\sqrt{1+x}+\sqrt[3]{1+x}} = \int \frac{6t^5\,dt}{t^3+t^2} = 6\int \frac{t^3\,dt}{t+1}$$

$$= 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C$$

$$= 2\sqrt{1+x} - 3\sqrt[3]{1+x} + 6\sqrt[6]{1+x} - 6\ln\left(\sqrt[6]{1+x}+1\right) + C.$$

The integral $\int R[x, (x-a)^{p/n}, (x-b)^{q/n}]\,dx$, where R is a rational function and p, q, n are integers, is an elementary function, if $p + q = kn$, where k is an integer.

2°. The integral $\int R(x, y)\,dx$, where R is a rational function of x and $y = \sqrt{ax^2 + bx + c}$ can be reduced to an integral of a rational fraction by means of one of the following substitution *(Euler's substitutions)*:

$$y = t \pm x\sqrt{a} \quad (a > 0), \tag{5.19}$$

$$y = tx \pm \sqrt{c} \quad (c > 0), \tag{5.20}$$

$$y = t(x - x_1) \quad (4ac - b^2 < 0), \tag{5.21}$$

where x_1 is one of the roots of the trinomial $ax^2 + bx + c$.

The integral $\int R(x, \sqrt{ax^2 + bx + c})\, dx$ can be brought to the form $\int R_1(\sin t, \cos t)\, dt$, where R_1 is also a rational function of $\sin t$ and $\cos t$, by means of the trigonometric substitutions

$$x + \frac{b}{2a} = \begin{cases} \dfrac{\sqrt{b^2 - 4ac}}{2a} \sin t, \\ \dfrac{\sqrt{b^2 - 4ac}}{2a} \cos t \end{cases} \quad (a < 0, \quad 4ac - b^2 < 0); \tag{5.22}$$

$$x + \frac{b}{2a} = \begin{cases} \dfrac{\sqrt{b^2 - 4ac}}{2a} \sec t, \\ \dfrac{\sqrt{b^2 - 4ac}}{2a} \operatorname{cosec} t \end{cases} \quad (a > 0, \quad 4ac - b^2 < 0); \tag{5.23}$$

$$x + \frac{b}{2a} = \begin{cases} \dfrac{\sqrt{4ac - b^2}}{2a} \tan t, \\ \dfrac{\sqrt{4ac - b^2}}{2a} \cot t \end{cases} \quad (a > 0, \quad 4ac - b^2 > 0), \tag{5.24}$$

Euler's substitutions and trigonometric substitutions sometimes lead to complicated computations. Therefore, in order to calculate the integral $\int R(x, y)\, dx$, where $y = \sqrt{ax^2 + bx + c}$ it is possible to apply the method of the expansion of the function under the integral sign into a sum of terms.

A rational function $R(x, y)$ can always be represented in the form

$$R(x, y) = \frac{P_1(x) + P_2(x)y}{P_3(x) + P_4(x)y}, \tag{5.25}$$

where P_i $(i = 1, 2, 3, 4)$ are integral polynomials. On multiplying the numerator and the denominator of the fraction by $P_3(x) - P_4(x)y$ we get

$$R(x, y) = R_1(x) + R_2(x)y,$$

where $R_1(x)$ and $R_2(x)$ are rational functions of x.

The integral of the first term is taken in its finite form. We multiply and divide the second term by y; we have the function

$$\frac{R_3(x)}{\sqrt{ax^2 + bx + c}}. \tag{5.26}$$

We separate the integral part $P_n(x)$ of the rational function $R_{?}(x)$, and we resolve the remaining regular fraction into partial fractions. Then, the integral of the second term is reduced to integrals of the following three types:

$$\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}}\, dx, \tag{5.27}$$

$$\int \frac{dx}{(x - x_1)^k \sqrt{ax^2 + bx + c}}, \tag{5.28}$$

$$\int \frac{Mx + N}{(x^2 + px + q)^k \sqrt{ax^2 + bx + c}}\, dx, \tag{5.29}$$

where all coefficients are real and the roots of the trinomial $x^2 + px + q$ are imaginary.

The integral (5.27) is taken by means of the method of indeterminate coefficients, according to the formula

$$\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}}\, dx$$

$$= Q_{n-1}(x) \sqrt{ax^2 + bx + c} + \lambda \cdot \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \tag{5.30}$$

where

$$Q_{n-1}(x) = A_1 x^{n-1} + A_2 x^{n-2} + A_3 x^{n-3} + \cdots + A_{n-1} x + A_n.$$

On differentiating both sides of the identity with respect to x and multiplying both sides of the identity obtained by $\sqrt{ax^2 + bx + c}$, we get

$$P_n(x) = Q'_{n-1}(x)\,(ax^2 + bx + c) + \tfrac{1}{2}\, Q_{n-1}(x)\,(2ax + b) + \lambda.$$

Both sides of this identity contain polynomials. Comparing the coefficients of equal powers of x we arrive at a set of $n + 1$ linear equations, from which we determine the coefficients of the polynomial $Q_{n-1}(x)$ and the constant λ.

Thus, the separation of the algebraic part of the integral $\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}}\, dx$ has been carried out. We still have to work

out the integral $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, considered earlier. The integral (5.28) becomes an integral of type (5.27) by the substitution $x - x_1 = 1/t$.

In the case in which $ax^2 + bx + c$ differs from $x^2 + px + q$ in the constant factor only, the integral (5.29) takes the form

$$\int \frac{Mx + N}{(ax^2 + bx + c)^{\frac{2k+1}{2}}}\, dx \tag{5.31}$$

and is represented by the sum of two integrals

$$\frac{M}{2a} \int \frac{2ax + b}{(ax^2 + bx + c)^{\frac{2k+1}{2}}}\, dx + \left(N - \frac{Mb}{2a}\right) \int \frac{dx}{(ax^2 + bx + c)^{\frac{2k+1}{2}}}, \tag{5.32}$$

the first of which is evaluated directly, by the substitution $t = ax^2 + bx + c$ and the second one is reduced to the integral of a polynomial by means of *Abel's substitution*

$$t = \left(\sqrt{ax^2 + bx + c}\right)' = \frac{ax + \frac{b}{2}}{\sqrt{ax^2 + bx + c}}.$$

In a more general case, it is possible to annihilate the terms of the first power in the integrand by means of a linear-rational substitution $x = (\alpha t + \beta)/(t + 1)$, where α and β are indeterminate coefficients, for $b/a \neq p$ and by means of a linear substitution $x = t - p/2$, for $b/a = p$, and to obtain a sum of integrals of the form

$$\int \frac{At + B}{(t^2 + \lambda)^m \sqrt{\alpha t^2 + \beta}}\, dt \quad (m = 1, 2, \ldots, k). \tag{5.33}$$

Each of these integrals resolves into two:

$$\frac{A}{\alpha} \int \frac{\alpha t\, dt}{(t^2 + \lambda)^m \sqrt{\alpha t^2 + \beta}} + B \int \frac{dt}{(t^2 + \lambda)^m \sqrt{\alpha t^2 + \beta}},$$

the first of which is worked out by means of a substitution $u = \sqrt{\alpha t^2 + \beta}$ and the second one by means of Abel's substitution $u = \alpha t/\sqrt{\alpha t^2 + \beta}$.

Thus, integrals of the type $\int R(x, \sqrt{ax^2 + bx + c})\, dx$ are always taken in the finite form and are expressed by means of the same functions as the integrals of rational functions and also by means of square roots.

In particular, the integral $\int \frac{\alpha x^2 + \beta x + c}{\sqrt{ax^2 + bx + c}}\, dx$ represents an algebraic function, if the condition

$$4a(c\alpha + b\beta) = 8a^2\gamma + 3b^2\alpha \quad (a \neq 0) \tag{5.34}$$

is satisfied.

3°. The integral $\int x^m(a + bx^n)^p dx$, where m, n, p are rational numbers *(the integral of the binomial differential)* is expressible in terms of elementary functions only if one of the following conditions is fulfilled *(Tchebyshev's conditions for the integrability of a binomial differential):* (a) if p is an integer, (b) if $(m + 1)/n$ is an integer, (c) if $(m + 1)/n + p$ is an integer.

In the case (a) for $p > 0$ we obtain a sum of power integrals; for $p < 0$, a substitution $x = z^N$, where N is the common denominator of fractions m and n, leads to an integral of a rational function.

In the case (b) we put $a + bx^n = z^N$, where N is the denominator of the fraction p.

In the case (c) the substitution $a + bx^n = z^N x^n$, where N is the denominator of fraction p, is applied.

Example 5. Calculate the integral

$$\int \sqrt[3]{x}\, \sqrt[3]{1 + 3\sqrt[3]{x^2}}\, dx.$$

Here $(m + 1)/n = 2$, and it follows that we put

$$1 + 3\sqrt[3]{x^2} = z^3,$$

whence

$$x = \frac{(z^3 - 1)^{3/2}}{3\sqrt{3}}, \quad dx = \frac{\sqrt{3}}{2}(z^3 - 1)^{1/2} z^2\, dz$$

and

$$\int \sqrt[3]{x}\, \sqrt[3]{1 + 3\sqrt[3]{x^2}}\, dx = \frac{1}{2}\int z^3(z^3 - 1)\, dz = \frac{z^7}{14} - \frac{z^4}{8} + C$$

$$= \frac{\left(1 + 3\sqrt[3]{x^2}\right)^{7/3}}{14} - \frac{\left(1 + 3\sqrt[3]{x^2}\right)^{4/3}}{8} + C.$$

EXAMPLE 6. Calculate the integral

$$\int \frac{dx}{x^{11}\sqrt{1+x^4}}.$$

Here $(m+1)/n + p = -3$; putting

$$1 + x^4 = z^2 x^4,$$

we have

$$x = \frac{1}{(z^2-1)^{1/4}}, \quad dx = -\frac{z\,dz}{2(z^2-1)^{5/4}}$$

and

$$\int \frac{dx}{x^{11}\sqrt{1+x^4}} = -\frac{1}{2}\int (z^2-1)^2\,dz = -\frac{z^5}{10} + \frac{z^3}{3} - \frac{z}{2} + C$$

$$= -\frac{1}{10}\frac{\sqrt{(1+x^4)^5}}{x^{10}} + \frac{1}{3}\frac{\sqrt{(1+x^4)^3}}{x^6} - \frac{1}{2}\frac{\sqrt{1+x^4}}{x^2} + C.$$

The integral $\int \sqrt{1+x^m}\,dx$, where m is a rational number, represents an elementary function, if $m = 2/k$, where $k = \pm 1, \pm 2, \ldots$

4. A general theory of integration of other elementary functions does not exist. Separate cases are known when an integral may be transformed into an integral of a rational or algebraic function by means of an appropriate substitution. However, the number of classes of elementary functions, whose integrals are also elementary, is very limited. Such integrals as, for example,

$$\int f(x, e^x)\,dx, \quad \int f(x, \ln x)\,dx, \quad \int f(x, \sin x, \cos x)\,dx,$$
$$\int f(e^x, \sin x, \cos x)\,dx,$$

where f is an algebraic, or even a rational function, represent, generally, new non-elementary functions.

The integral $\int R(x)\, e^{ax}\,dx$, where R is a rational function, whose denominator has real roots only, is expressible in terms of elementary functions and the non-elementary function

$$\int \frac{e^{ax}}{x}\,dx = \mathrm{li}\,(e^{ax}) + C, \tag{5.35}$$

where

$$\mathrm{li}\,(x) = \int \frac{dx}{\ln x}. \tag{5.36}$$

The integral $\int P(1/x)\, e^x\, dx$, where $P(1/x) = a_0 + a_1/x + a_2/x^2 + \dots + a_n/x^n$, and $a_0, a_1, a_2, \dots, a_n$ are constants, represents an elementary function, if the condition $a_1 + a_2/1! + a_3/2! + \dots + a^n/(n-1)! = 0$ is satisfied.

Some Particular Cases

1°. The integral

$$\int F(e^{ax}, e^{bx}, \dots, e^{kx})\, dx, \tag{5.37}$$

where F is an algebraic function, and $a, b, \dots, k$ are commensurate numbers, can be reduced to an integral of an algebraic function. In particular, the integral

$$\int R(e^{ax}, e^{bx}, \dots, e^{kx})\, dx, \tag{5.38}$$

where R is a rational function, is always elementary. By means of the substitution $x = ay$, it is reduced to the integral $\int R_1(e^y)\, dy$ which is transformed into an integral of a rational function by means of a new substitution $e^y = z$. Since $\sinh x$ and $\cosh x$ are rational functions of e^{ix}, it follows, hence, that integrals

$$\int R(\sinh x, \cosh x)\, dx, \quad \int R(\sin x, \cos x)\, dx \tag{5.39}$$

represent elementary functions. For the second of these integrals the substitution given above is imaginary; it is more convenient to use in its case, the so-called *universal substitution*

$$\tan \frac{x}{2} = t, \tag{5.40}$$

also leading to an integral of a rational function, since

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2\, dt}{1+t^2}. \tag{5.41}$$

An analogous substitution

$$\tanh \frac{x}{2} = t \tag{5.42}$$

can be applied also to the first integral. Here

$$\sinh x = \frac{2t}{1-t^2}, \quad \cosh t = \frac{1+t^2}{1-t^2}, \quad dx = \frac{2\, dt}{1-t^2}. \tag{5.43}$$

If the equality

$$R(\sin x, -\cos x) = -R(\sin x, \cos x)$$

or

$$R(-\sin x, \cos x) = -R(\sin x, \cos x)$$

holds, it is useful to apply the substitutions

$$\sin x = t \tag{5.44}$$

or

$$\cos x = t, \tag{5.45}$$

respectively.

If the equation

$$R(-\sin x, -\cos x) = R(\sin x, \cos x)$$

holds, the simplest substitution to use is

$$\tan x = t. \tag{5.46}$$

The same occurs also in connection with the first integral.

Integrals

$$\int R(\sinh x, \cosh x, \sinh 2x, \ldots, \cosh mx)\, dx, \tag{5.47}$$

$$\int R(\sin x, \cos x, \sin 2x, \ldots, \cos mx)\, dx \tag{5.48}$$

belong to the two types indicated.

2°. The integral

$$\int P(x, e^{ax}, e^{bx}, \ldots, e^{kx})\, dx, \tag{5.49}$$

where $a, b, \ldots, k$ are arbitrary numbers and P is a polynomial, is always an elementary function, since it can be represented in the form of a sum of a finite number of integrals of the form $\int x^p e^{\alpha x}\, dx$, which are calculated according to the formula given on p. 137.

The following integrals can be reduced to that type:

$$\left.\begin{array}{ll} \int x^m(\sin px)^\mu (\cos qx)^\nu\, dx, & \int x^m(\sinh px)^\mu (\cosh qx)^\nu\, dx, \\ \int x^m e^{-ax}(\sin px)^\mu\, dx, & \int x^m e^{-ax}(\cos qx)^\nu\, dx, \end{array}\right\} \tag{5.50}$$

where m, μ and ν are integers.

In the same way, the integrals

$$\int P(x, \ln x)\, dx, \quad \int P(x, \arcsin x)\, dx, \quad \text{and others,} \tag{5.51}$$

where P is a polynomial, can be reduced to the types of integrals shown above by means of the substitutions $x = e^y$, $x = \sin y$ and others.

3°. The integral

$$\int \sin^{\mu} x \cos^{\nu} x \, dx \tag{5.52}$$

is reducible to a sum of power integrals by the substitution $\sin x = z$, when μ is odd and greater than zero, and by the substitution $\cos x = z$ when ν is odd and greater than zero.

In the case when both μ and ν are even and positive, the degree can be lowered by applying the identities

$$\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}, \quad \cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2},$$

$$\sin x \cos x = \tfrac{1}{2} \sin 2x.$$

By means of the substitution $\cos x = y$ we can reduce the integral (5.52) to the integral of the binomial differential $\int y^{\mu}(1 - y^2)^{(\nu-1)/2} dx$. The integral is taken in the finite form only when μ or ν is a whole odd number, or when the sum $\mu + \nu$ is a whole even number or zero. Thus, for example $\int \sqrt{\sin x}\, dx$ is a transcendental function as is $\int \sqrt{\tan x}\, dx$.

4°. The integrals

$$\left.\begin{aligned} &\int \tan^{\mu} x \sec^{\nu} x \, dx, \quad \int \cot^{\mu} x \sec^{\nu} x \, dx, \\ &\int \tan^{\mu} x \operatorname{cosec}^{\nu} x \, dx, \quad \int \cot^{\mu} x \operatorname{cosec}^{\nu} x \, dx, \end{aligned}\right\} \tag{5.53}$$

where ν is positive and even and μ is arbitrary, is reducible to a sum of power integrals: the first two by means of the substitution $\tan x = z$, the second two by means of the substitution $\cot x = z$.

The integrals shown below can be calculated by the method of indeterminate coefficients, applying the following formulae:

$$\int \frac{\alpha \sin x + \beta \cos x}{a \sin x + b \cos x} dx = Ax + B \ln |a \sin x + b \cos x| + C_1, \tag{5.54}$$

$$\int \frac{\alpha \sin x + \beta \cos x + \gamma}{a \sin x + b \cos x + c} dx$$

$$= Ax + B \ln |a \sin x + b \cos x + c| + C \int \frac{dx}{a \sin x + b \cos x + c}, \tag{5.55}$$

$$\int \frac{\alpha \sin^2 x + 2\beta \sin x \cos x + \gamma \cos^2 x}{a \sin x + b \cos x}\, dx$$

$$= A \sin x + B \cos x + C \int \frac{dx}{a \sin x + b \cos x}, \tag{5.56}$$

$$\int \frac{\alpha \sin x + \beta \cos x}{a \sin^2 x + 2b \sin x \cos x + c \cos^2 x}\, dx$$

$$= A \int \frac{du_1}{k_1 u_1^2 + \lambda_1} + B \int \frac{du_2}{k_2 u_2^2 + \lambda_2}, \tag{5.57}$$

where A, B, C are indeterminate coefficients, λ_1, λ_2 are the roots of the equation

$$\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0 \quad (\lambda_1 \neq \lambda_2),$$

$$u_i = (a - \lambda_i) \sin x + b \cos x, \quad k_i = \frac{1}{a - \lambda_i} \quad (i = 1, 2).$$

Similarly

$$\int \frac{dx}{(a \sin x + b \cos x)^n} = \frac{A \sin x + B \cos x}{(a \sin x + b \cos x)^{n-1}}$$

$$+ C \int \frac{dx}{(a \sin x + b \cos x)^{n-1}}, \tag{5.58}$$

$$\int \frac{dx}{(a + b \cos x)^n} = \frac{A \sin x}{(a + b \cos x)^{n-1}} + B \int \frac{dx}{(a + b \cos x)^{n-1}}$$

$$+ C \int \frac{dx}{(a + b \cos x)^{n-2}} \quad (|a| \neq |b|). \tag{5.59}$$

§ 3. The Definite Integral

1. Suppose that the function $f(x)$ is defined in the interval $[a, b]$ $(a < b)$. Let us subdivide this interval into n elementary intervals by means of the points $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ and let us select an arbitrary point in each. The sum of products of the

values of the function in the points selected by the lengths of the corresponding subdivisions

$$\sigma_n = \sum_{k=1}^{n} f(\xi_k)\, \Delta_k x, \tag{5.60}$$

where ξ_k is the selected point of the interval $[x_{k-1}, x_k]$ and $\Delta_k x = x_k - x_{k-1}$ is the length of that interval, is called *the integral sum*

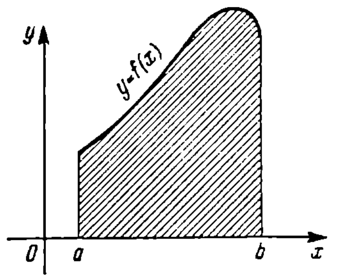

FIG. 10

(the Cauchy–Riemann sum) for the function $f(x)$ in the interval $[a, b]$.

If, on the unlimited increase of the number of subdivisions, where the length of each subdivision tends to zero, there exists a limit of the integral sums and it does not depend on the method of subdividing the interval $[a, b]$ or on the selection of points ξ_k, the function $f(x)$ is called *integrable in the interval* $[a, b]$. The limit of the integral sums

$$\lim_{\substack{n\to\infty \\ \max \Delta x \to 0}} \sum_{k=1}^{n} f(\xi_k)\, \Delta_k x = \int_a^b f(x)\, dx \tag{5.61}$$

is called *the definite integral of the function* $f(x)$ with respect to the interval $[a, b]$

This definition means that for any $\varepsilon > 0$, there can be found a $\delta > 0$ such that for any subdivision, whose elementary intervals are smaller in length than δ, i.e. $\max \Delta_k x < \delta$, and for any choice of points ξ_k the following inequality holds

$$\left| \sum_{k=1}^{n} f(\xi_k)\, \Delta_k x - \int_a^b f(x)\, dx \right| < \varepsilon .$$

Geometrically, the definite integral, for $f(x) > 0$, expresses the area under the curve (Fig. 10).

If a function $f(x)$ is continuous in the interval $[a, b]$, *it can be integrated in that interval,* i.e. the definite integral $\int_a^b f(x)\,dx$ exists. A function can be integrated also if it is *bounded in the interval* $[a, b]$ *and has a finite number of discontinuities of the first kind (finite jumps) in it.*

The definite integral with respect to the interval $[a, b]$ can also be defined for $a > b$. Here, the differences $\Delta_k x = x_{k+1} - x_k$, contained in the integral sum, are the lengths of the corresponding subdivisions but are of negative sign.

2. The definite integral of the functions $f(x)$ and $\varphi(x)$ to be integrated possesses the following properties:

(a)
$$\int_a^a f(x)\,dx = 0. \tag{5.62}$$

(b)
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{5.63}$$

(c)
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \quad (k = \text{const}). \tag{5.64}$$

(d)
$$\int_a^b [f(x) \pm \varphi(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b \varphi(x)\,dx. \tag{5.65}$$

(e)
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{5.66}$$

(f) *If* $a < b$ *and* $f(x) \geqslant 0$ *then* $\int_a^b f(x)\,dx \geqslant 0$; *if* $f(x) > 0$, *then* $\int_a^b f(x)\,dx \geqslant 0$; *for continuous functions, it follows from* $\int_a^b f(x)\,dx = 0$ *and* $f(x) \geqslant 0$ *that* $f(x) \equiv 0$.

(g) *If* $a < b$ *and* $f(x) \geqslant \varphi(x)$ *then*

$$\int_a^b f(x)\,dx \geqslant \int_a^b \varphi(x)\,dx. \tag{5.67}$$

(h)
$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx. \tag{5.68}$$

These properties are used in calculations and in estimates of the values of definite integrals. Frequent use is also made of the *mean value theorem.*

THEOREM 2 (THE FIRST MEAN VALUE THEOREM). *If the function $f(x)$ is continuous in the interval $[a, b]$ $(a < b)$ and the function $\varphi(x)$ can be integrated and is of constant sign in $[a, b]$, then there exists in the interval (a, b) a point ξ, such that*

$$\int_a^b f(x)\,\varphi(x)\,dx = f(\xi)\int_a^b \varphi(x)\,dx \quad (a < \xi < b). \tag{5.69}$$

In particular, for $\varphi(x) \equiv 1$, the first mean value theorem gives for the continuous function $f(x)$

$$\int_a^b f(x)\,dx = f(\xi)\,(b - a) \quad (a < \xi < b). \tag{5.70}$$

The value

$$f(\xi) = \frac{1}{b - a}\int_a^b f(x)\,dx \tag{5.71}$$

is called *the mean value of the function $f(x)$ in the interval* $[a, b]$. Geometrically, the mean value of a function equals the altitude of a rectangle equal in area to the area under the curve and having a common base with it. (Cf. Fig. 11.)

THEOREM 3 (THE SECOND MEAN VALUE THEOREM). *If the functions $f(x)$ and $\varphi(x)$ are bounded and can be integrated in $[a, b]$, $a < b$, and*

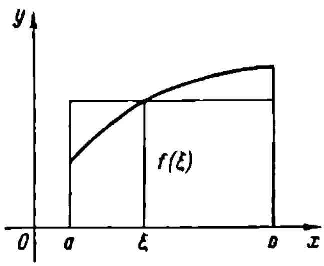

FIG. 11

the function $\varphi(x)$ satisfies the inequalities $A \leqslant \varphi(x) \leqslant B$ in $[a, b]$ and does not decrease, then there exists a point ξ in the interval $[a, b]$, such that

$$\int_a^b f(x)\,\varphi(x)\,dx = A\int_a^\xi f(x)\,dx + B\int_\xi^b f(x)\,dx. \tag{5.72}$$

If the function $\varphi(x)$ does not increase, then

$$\int_a^b f(x)\,\varphi(x)\,dx = B\int_a^\xi f(x)\,dx + A\int_\xi^b f(x)\,dx. \qquad (5.73)$$

In particular, in these formulae we may put $A = \varphi(a + 0)$, $B = \varphi(b - 0)$, i.e. for a non-decreasing function $\varphi(x)$ we have:

$$\int_a^b f(x)\,\varphi(x)\,dx = \varphi(a + 0)\int_a^\xi f(x)\,dx + \varphi(b - 0)\int_\xi^b f(x)\,dx. \qquad (5.74)$$

The same kind of formula holds for the case when $\varphi(x)$ does not increase, since in this case the converse can be taken:

$$A = \varphi(b - 0), \quad B = \varphi(a + 0).$$

If the function $\varphi(x)$ is strictly monotonic, then

$$\int_a^b f(x)\,\varphi(x)\,dx = \varphi(a)\int_a^\xi f(x)\,dx + \varphi(b)\int_\xi^b f(x)\,dx. \qquad (5.75)$$

3. The indefinite integral, considered in § 1, is a function of the variable of integration x. On the contrary, the definite integral of a given function with respect to a given interval is a fixed number, independent of the variable of integration. Therefore, the variable of integration can be denoted by any letter

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt.$$

If the function $f(x)$ is continuous on the interval $[a, b]$ and x is any point in the interval, the integral $\int_a^x f(t)\,dt$ exists (the variable of integration is denoted by another letter, to distinguish it from the upper limit of integration), is completely defined by the choice of point x, and alters when x is altered. Thus, *the definite integral is a function of its upper limit:*

$$F(x) = \int_a^x f(t)\,dt. \qquad (5.76)$$

The definite integral is a continuous and differentiable function of its upper limit and

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x), \tag{5.77}$$

$$d\int_a^x f(t)\,dt = f(x)\,dx \tag{5.78}$$

at all points of continuity of $f(x)$, *i.e. the definite integral, as a function of its upper limit, is one of the primitives for the function under the sign of the integral.* Namely it is that primitive which becomes zero for the value of x equal to the lower limit of integration.

Since all primitives differ only by the constant term, therefore, if we denote an arbitrary original of the function $f(x)$ by $\Phi(x)$, we find

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a) = \Phi(x)\Big|_a^b \tag{5.79}$$

(the Newton–Leibniz formula), i.e. the definite integral equals the increment in the interval of integration of the primitive of the function being integrated.

The Newton–Leibniz formula is the basic tool for the calculation of definite integrals; it reduces the evaluation of these integrals to finding an indefinite integral (i.e. a primitive) of the relevant function. In the case when, for the purpose of finding the primitive, the indefinite integral has to be transformed in some sort of way, use is made of formulae for *substitution (change of variable)* or of *integration by parts* for the definite integral.

THE FORMULA FOR SUBSTITUTION: Suppose the function $f(x)$ is continuous in $[a, b]$ and the function $x = \varphi(t)$ is of constant sign and is continuously differentiable in $[\alpha, \beta]$, and $a \leqslant \varphi(t) \leqslant b$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$. Then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt. \tag{5.80}$$

THE FORMULA FOR INTEGRATION BY PARTS

$$\int_a^b f(x)\,\varphi'(x)\,dx = f(b)\,\varphi(b) - f(a)\,\varphi(a) - \int_a^b \varphi(x) f'(x)\,dx \tag{5.81}$$

or

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du. \tag{5.82}$$

THE FORMULA FOR SUCCESSIVE INTEGRATION BY PARTS:

$$\int_a^b uv^{(n)}dx = [uv^{(n-1)} + u'v^{(n-2)} + u''v^{(n-3)} + \cdots$$
$$+ (-1)^{n-1}u^{(n-1)}v]_a^b - (-1)^n \int_a^b u^{(n)}v\,dx. \tag{5.83}$$

CAUCHY FORMULA:

$$\int_{x_0}^x dx_1 \int_{x_0}^{x_1} dx_2 \ldots \int_{x_0}^{x_{n-1}} f(x_n)\,dx_n = \frac{1}{(n-1)!}\int_x^x (x-t)^{n-1} f(t)\,dt. \tag{5.84}$$

Note. If we assign a number $J[y(x)]$ to each function $y = y(x)$ of some system of functions M, it is said that we are given *a functional* J in M. Here, if J satisfies the following conditions:

$$(1)\quad J[y_1 + y_2] = J(y_1 + y_2]; \quad (2)\quad J[\lambda y] = \lambda J[y],$$

where λ is any real number, then $J[y]$ is called *a linear functional* in M.

The definite integral $J[y] = \int_a^b y\,dx$ is a linear functional in the class of functions capable of integration, $y = y(x)$, defined in $[a, b]$.

4. Suppose that the function $y = f(x)$ is defined and is bounded in the interval $[a, b]$. Then it has an upper and a lower bound in the whole interval as well as in any part of it. Let us subdivide the interval $[a, b]$ into elementary segments. We denote the length of the kth elementary segment by $\Delta_k x$, and the upper and lower boundaries of the function in this segment by M_k and m_k respectively.

The names "upper" and "lower integral sum" (Darboux sums) are given to the sums of products of upper and lower bounds of the function in the elementary segments by the length of the latter respectively:

$$S_n = \sum_{k=1}^n M_k \Delta_k x, \tag{5.85}$$

$$s_n = \sum_{k=1}^n m_k \Delta_k x. \tag{5.86}$$

For any subdivision of the interval, the Darboux sums and the Cauchy–Riemann sums are connected by the following relationship

$$\sum_{k=1}^{n} m_k \Delta_k x \leqslant \sum_{k=1}^{n} f(\xi_k)\Delta_k x \leqslant \sum_{k=1}^{n} M_k \Delta_k x. \tag{5.87}$$

The upper integral I^* of the function $f(x)$ in the interval $[a, b]$ is the name of the limit† of the upper integral sums

$$I^* = \lim_{\substack{n\to\infty \\ \max \Delta_k x \to 0}} \sum_{k=1}^{n} M_k \Delta_k x. \tag{5.88}$$

Similarly, *the lower integral* I_* is the name of the limit of lower integral sums

$$I_* = \lim_{\substack{n\to\infty \\ \max \Delta_k x \to 0}} \sum_{k=1}^{n} m_k \Delta_k x. \tag{5.89}$$

The existence of the upper and lower integrals is established by the following theorem.

THEOREM 4 *(Darboux theorem). If the function $f(x)$ is bounded in the interval $[a, b]$, then its lower and upper integral sums have a limit, i.e. any bounded function possesses a lower and an upper integral.*

The proof of the Darboux theorem is based on two important properties of upper and lower sums:

(a) In changing from a given subdivision to a new one, which is obtained by adding new points of subdivision, the lower integral sum does not decrease and the upper one does not increase;

(b) The lower integral sum for any subdivision is not greater than the upper integral sum for any other subdivision.

5. Upper and lower integrals, defined in § 4, can have different values, or they can coincide. If the upper and the lower integrals of the function $f(x)$ with respect to the interval $[a, b]$ coincide, their common value is called *the Riemann integral of the function $f(x)$ with respect to this interval* and we write

$$(R)\int_a^b f(x)\,dx = I^* = I_*. \tag{5.90}$$

Functions $f(x)$ for which $(R)\int_a^b f(x)\,dx$ exists are said to be *Riemann-integrable.*

† The limits of integrals are understood in the sense indicated in sec. 1.

The *condition of integrability* which is simplest to formulate is the following: the function $f(x)$ is Riemann-integrable in the interval $[a, b]$ if, and only if, it is bounded and

$$\lim_{\max \Delta_k x \to 0} (S_n - s_n) = 0, \tag{5.91}$$

i.e. the limit of the difference of the upper and lower integral sums equals zero.

The boundedness of a function is insufficient to ensure that it is Riemann-integrable. Thus, the function, defined in the interval $[0, 1]$ by the equation

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number, } x = p/q, \\ 0, & \text{if } x \text{ is irrational} \end{cases} \tag{5.92}$$

(Dirichlet function), is bounded. However, for any segment in any subdivision we have $M_k = 1$, while $m_k = 0$. As a result of this, the upper sum always equals

$$S_n = \sum_{k=1}^{n} M_k \Delta_k x = \sum_{k=1}^{n} \Delta_k x = 1,$$

and the lower sum

$$s_n = \sum_{k=1}^{n} m_k \Delta_k x = 0.$$

Thus, for the Dirichlet function $I^* = 1$, $I_* = 0$ and the Riemann integral does not exist.

The class of functions which are Riemann-integrable includes functions continuous in a given interval, and, more generally, functions which have in that interval a finite number of discontinuities of the first kind, and also bounded monotonic functions and, more generally, functions of bounded variation.

Since Cauchy–Riemann sums are enclosed between Darboux sums, the Riemann integral is the common limit of the integral sums of Darboux and of Cauchy–Riemann. Therefore the Riemann integral coincides with the usual definite integral considered in sec. 1, so that the sign (R) in front of the integral is usually omitted. It also follows that the properties of the Riemann integral coincide with the properties of the definite integral considered in sec. 1.

Darboux sums may be constructed also for a function which is not necessarily defined at every point of the interval, while for Cauchy sums such an assumption is necessary. Therefore it can

be formally reckoned that the class of functions which are Riemann-integrable is wider than that of those integrable in the sense of the definition in section 1.

6. A numerical sequence $\{x_n\}$, all of whose points belong to the interval [0, 1], is said to be *uniformly distributed* in the interval if the number of points of the sequence with suffixes smaller than a given n, which find themselves in the segment $[\alpha, \beta]$ of the interval [0, 1] at its limit, is proportional to the length of the interval. More exactly, *if $\nu_n(\alpha, \beta)$ denotes the number of points of the sequence with suffixes smaller than n belonging to the segment* $[\alpha, \beta]$, *then*

$$\lim_{n\to\infty} \frac{\nu_n(\alpha, \beta)}{n} = \beta - \alpha. \tag{5.93}$$

In the terms of the theory of probability this definition means that *the probability of a random element of the sequence falling on the segment* $[\alpha,\beta]$ *equals the length of this segment.* (See volume 69 of this series.)

The simplest example of a uniformly distributed sequence is the sequence of all proper vulgar fractions, namely, of fractions of form m/n ($n = 2, 3, \ldots$; $m = 1, 2, \ldots, n - 1$). This sequence has the form

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \ldots$$

A great number of examples of uniformly distributed sequences can be constructed with the help of the following theorem:

THEOREM 5. *If ϑ is irrational, then the sequence $x_n = n\vartheta - [n\vartheta]$ ($n = 1, 2, \ldots$) is uniformly distributed in the interval* [0, 1], *where the symbol $[n\vartheta]$ denotes the integral part of the number $n\vartheta$, i.e. the greatest whole number not exceeding $n\vartheta$.*

Sequences, which are uniformly distributed in an interval, can be utilized in the calculation of integrals. Their role in working out integrals is based on the following theorem.

THEOREM 6. *If the numerical sequence $\{x_n\}$ is evenly distributed in the inteval* [0, 1], *then for any function $f(x)$ integrable in* [0, 1] *the following asymptotic relation holds*

$$\lim_{n\to\infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \int_0^1 f(x)\, dx. \tag{5.94}$$

Conversely, the consequence of the truth of this asymptotic relation is that the sequence $\{x_n\}$ is evenly distributed in the interval.

Thus, this relationship is a necessary and sufficient condition of the uniform distribution of the sequence, so that it can be regarded as the definition of uniform distribution.

As a corollary of the basic asymptotic relation it should be noted that for the sequence $\{x_n\}$ uniformly distributed in the interval [0, 1], for every positive integer k the following relation is satisfied

$$\lim_{n\to\infty} \frac{x_1^k + x_2^k + \cdots + x_n^k}{n} = \frac{1}{k+1}. \tag{5.95}$$

It can be obtained by putting $f(x) = x^k$.

A more general asymptotic relation also holds. If the sequence $\{x_n\}$ is uniformly distributed in the interval [0, 1] and $\{\alpha_n\}$ is a monotonically decreasing sequence of positive numbers with a divergent sum, then

$$\lim_{n\to\infty} \frac{\alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n)}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} = \int_0^1 f(x)\,dx. \tag{5.96}$$

§ 4. The Integration of Functions of n Variables

1. It seems possible, for functions with several variables, to define the different concepts of an integral in a way dependent on the number of dimensions of the domain of integration and on the number of arguments of the function. First, we consider functions with two variables.

A bounded closed region G of the plane xOy is called *squarable*, if the upper bound of the areas of polygons inscribed in it coincides with the lower bound of areas of polygons, circumscribed about $\overline{G}$. The general value Q of these bounds is called the *area* of the region. The *diameter* δ of the region $\overline{G}$ is the name given to the upper bound of distances between any two points of the region $\overline{G}$.

Suppose the function of a point of the region $z = f(P)$ or $z = f(x, y)$ is defined in the region $\overline{G}$ which is being squared. We subdivide the region $\overline{G}$ into a finite number of elementary squarable parts, whose areas are Δ_{kQ} $(k = 1, 2, \ldots, n)$. We select an arbitrary point (ξ_k, η_k) in each elementary area, and compile the *sum of*

products of the value of the function in selected points by the areas of the elementary subdivision:

$$\sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta_k q. \tag{5.97}$$

This sum is called *the (double) integral sum* for the function $f(x, y)$ in the region G. For a bounded function $f(x, y)$ it also is possible to construct *(double) upper and lower Darboux sums*

$$S_n = \sum_{k=1}^{n} M_k \Delta_k q, \quad s_n = \sum_{k=1}^{n} m_k \Delta_k q, \tag{5.98}$$

where M_k and m_k denote *the upper and lower bounds of the function* $f(x, y)$ respectively on the corresponding elementary areas.

The name *a double integral* $\int\int_G f(x, y)\, dq$ or $\int\int_G f(P)\, dq$ of the function $f(x, y)$ in the region G is applied to the limit of the integral sums, when the diameters of all elementary regions of subdivision tend to zero:

$$\int\int_G f(x, y)\, dq = \lim_{\substack{n\to\infty \\ \max \delta_k \to 0}} \sum_{k=1}^{n} f(\xi\ , \eta_k)\Delta_k q = \lim_{\substack{n\to\infty \\ \max \delta_k \to 0}} \sum_{k=1}^{n} M_k \Delta_k q$$

$$= \lim_{\substack{n\to\infty \\ \max \delta_k \to 0}} \sum_{k=1}^{n} m_k \;..\, J. \tag{5.99}$$

This means that for any $\varepsilon > 0$ there can be found a number $\eta > 0$, such that for any subdivision of the region G, for which the greatest diameter of an elementary region is smaller than η, $\max \delta_k < \eta$, the following inequality holds:

$$\left| \int\int_G f(x, y)\, dq - \sum_{k=1}^{u} f(\xi_k, \eta_k)\Delta_k q \right| < \varepsilon.$$

A similar inequality holds also for Darboux sums.

Sometimes, *the upper and lower integrals* are defined, as for a function of one variable, as limits of upper and lower Darboux sums respectively, and if they coincide their common value is called *the double Riemann integral.*

Double integrals possess properties analogous to those of the definite integral:

(a) $$\iint_G kf(x, y)\,dq = k\iint_G f(x, y)\,dq \quad (k = \text{const}). \qquad (5.100)$$

(b) $$\iint_G [f(x, y) \pm \varphi(x, y)]\,dq$$
$$= \iint_G f(x, y)\,dq \pm \iint_G \varphi(x, y)\,dq. \qquad (5.101)$$

(c) *If the region G is split up into parts G_1 and G_2, then*

$$\iint_G f(x, y)\,dq = \iint_{G_1} f(x, y)\,dq + \iint_{G_2} f(x, y)\,dq. \qquad (5.102)$$

(d) *If at all points of the region G the inequality $f(x, y) \geqslant \varphi(x, y)$ holds, then*

$$\iint_G f(x, y)\,dq \geqslant \iint_G \varphi(x, y)\,dq. \qquad (5.103)$$

(e) *If $f(x, y) \equiv 1$, then*

$$\iint_G dq = Q, \qquad (5.104)$$

where Q denotes the area of the region G.

(f) *If at all points of the region G, the inequalities $m \leqslant f(x, y) \leqslant M$ hold, then*

$$mQ \leqslant \iint_G f(x, y)\,dq \leqslant MQ. \qquad (5.105)$$

(g) *If $f(x, y)$ is continuous in the region G, including the boundary, there exists a point (ξ, η) lying inside G, for which*

$$\iint_G f(x, y)\,dq = f(\xi, \eta)Q$$

(the mean value theorem).

The value of $f(\xi, \eta)$, defined by the latter equation, is called the *mean value* of the function $f(x, y)$ in the region G.

The existence of a double integral can be guaranteed for the classes of functions indicated in the following theorems, where *the region of integration is always assumed to be bounded and squarable.*

THEOREM 7. *The double integral of a continuous function exists.*

THEOREM 8. *If the function $f(x, y)$ is bounded and its points of discontinuity lie on a finite number of curves, which are graphs of the continuous functions $y = \varphi(x)$ or $x = \varphi(y)$ then it is integrable.*

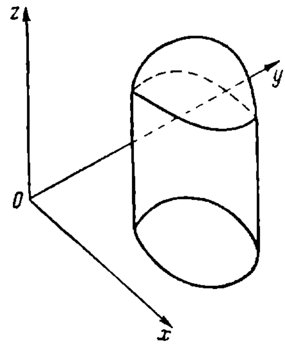

FIG. 12

This theorem is a particular case of the following more general theorem.

THEOREM 9. *If the function $f(x, y)$ is bounded and the set of points of discontinuity has zero area, then there exists a double integral of such a function.*

Geometrically, the double integral of a positive function expresses the volume of a body, which is customarily known as a *cylindroid* (Fig. 12). In order to calculate double integrals they are reduced to repeated ones; these are usually written down in the form $\int\int_G f(x, y)\, dx\, dy$.

The name *repeated* (or *twofold*) *integral*

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \quad \text{or} \quad \int_\alpha^\beta dy \int_{\psi_1(y)}^{\varphi_2(y)} f(x, y)\, dx$$

is given to a finite integral of a definite integral

$$\int_a^b \Phi(x)\, dx, \quad \text{where} \quad \Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy, \qquad (5.106)$$

or

$$\int_\alpha^\beta \Psi(y)\, dy, \quad \text{where} \quad \Psi(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx. \qquad (5.107)$$

In order to find a repeated integral, the *inner integral* should first be calculated, by integrating the function $f(x, y)$ in y (or in x) and then one should integrate the obtained function $\Phi(x)$ (or $\Psi(y)$).

If the region G is bounded by continuous curves $y = \varphi_1(x)$ and $y = \varphi_2(x)$ $(a \leqslant x \leqslant b, \varphi_1(x) \leqslant \varphi_2(x))$ and by the ordinates $x = a$, $x = b$ (Fig. 13), and there exist the double integral $\int\int_G f(x, y)\,dx\,dy$ and the definite integral $\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy$ for any x in a, b, then there also exists the repeated integral $\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy$ and the following equation holds:

$$\int\int_G f(x, y)\,dx\,dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy. \qquad (5.108)$$

Similarly, *if the region G is bounded by continuous curves $x = \psi_1(y)$ and $x = \psi_2(y)$ $(\alpha \leqslant y \leqslant \beta,\ \psi_1(y) \leqslant \psi_2(y))$ (Fig. 14) and by the straight lines $y = \alpha$, $y = \beta$, then, in the same conditions*

$$\int\int_G f(x, y)\,dx\,dy = \int_\alpha^\beta dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx. \qquad (5.109)$$

If the region G has more complicated boundaries, then, in order to reduce a double integral to a repeated one, the region should be

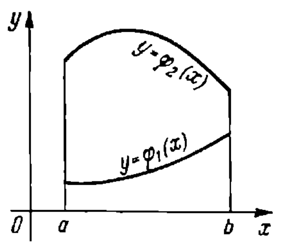

FIG. 13

subdivided into parts, whose shape is depicted in Fig. 13 or Fig. 14.

Suppose the function $f(x, y)$ is defined in the region G and the image of the region Δ of the plane u, v into the region G is given by means of the functions

$$x = x(u, v), \quad y = y(u, v).$$

Then the double integral in the region G is transformed according to the formula

$$\iint_G f(x, y)\, dx\, dy = \iint_\Delta f[x(u, v), y(u, v)] \left| \frac{D(x, y)}{D(u, v)} \right| du\, dv, \tag{5.110}$$

which is called the formula of *change of variable in a double integral.* Just as in Chapter III, $\partial(x, y)/\partial(u, v)$ denotes the Jacobian of the

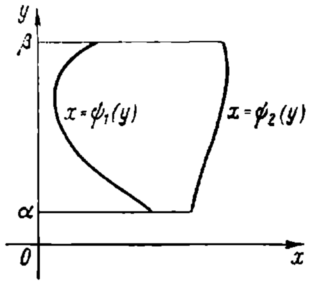

FIG. 14

mapping. In particular, in changing to polar coordinates the formula of change of variable takes the form

$$\iint_G f(x, y)\, dx\, dy = \iint_G f(\varrho \cos \varphi, \varrho \sin \varphi) \varrho\, d\varrho\, d\varphi. \tag{5.111}$$

The transformation of double integrals is considered in greater detail in Chapter VII.

2. The volume of a three-dimensional region V can be determined with the help of the volumes of inscribed and circumscribed polyhedra, in a way similar to that used in sec. 1 for area. Regions having volume are called *cubable.*

For the function $f(x, y, z)$ defined in a cubable region, a (triple) integral Cauchy–Riemann sum is constructed:

$$\sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) \Delta_k v. \tag{5.112}$$

When the function is bounded, then, in addition to the Cauchy–Riemann sum, it is also possible to construct *upper and lower Darboux sums* (see sec. 1).

If the integral sums have a common limit when the diameters of all elementary regions tend to zero – a limit which does not depend on methods of subdivision and selection of points (ξ_k, η_k, ζ_k) (for the Cauchy–Riemann sum), then this limit is called ***a triple integral*** of the function $f(x, y, z)$ in the given region V. It is written down in the symbols

$$\iiint_V f(x, y, z)\, dv, \quad \iiint_V f(P)\, dv$$

or

$$\iiint_V f(x, y, z)\, dx\, dy\, dz. \tag{5.113}$$

In order to calculate the triple integral it is reduced to a threefold one, which is found by means of three successive integrations with respect to each separate variable:

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x, y, z)\, dz. \tag{5.114}$$

Just as for double integrals, *the formula of change of variable* in the triple integral has the form

$$\begin{aligned} &\iiint_V f(x, y, z)\, dx\, dy\, dz \\ &= \iiint_\Omega f[x(u, v, w), \quad y(u, v, w), \quad z(u, v, w)] \times \\ &\times \left| \frac{D(x, y, z)}{D(u, v, w)} \right| du\, dv\, dw, \end{aligned} \tag{5.115}$$

if the region Ω is mapped into the region V by means of functions

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w).$$

3. Suppose that in n-dimensional space a certain *measure* is defined, which sets up a correspondence between certain n-dimensional regions and definite positive numbers – n-dimensional volumes. For functions defined in these measurable (i.e. having an n-dimensional volume) n-dimensional regions, it is possible to define Cauchy–Riemann's integral sums, and the integral as the limit of these sums.

If the function is bounded, then, in addition to Cauchy–Riemann sums, it is possible also to construct upper and lower Darboux sums and to define the n-dimensional Riemann integral as the common limit of the upper and lower Darboux sums.

The whole theory of double and triple integrals extends almost automatically to include the n-dimensional case.

4. In secs. 1 to 3 we considered the integration of functions of n variables ($n = 2, 3, \ldots$) in a region of n dimensions. It is possible, however, to have integration in manifolds of a smaller number of dimensions. Let us examine particular cases.

Let the function $f(x, y)$ be defined in the points of a plane curve L. Let us subdivide the curve into m segments and let us select in each of them a point (ξ_k, η_k). Then it is possible to construct the *integral sum*

$$\sum_{k=1}^{m} f(\xi_k, \eta_k)\, \Delta_k s, \tag{5.116}$$

where $\Delta_k s$ denotes the length of the corresponding segment of the curve, whose limit, when the lengths of all segments of subdivision tend to zero, is called *the integral over a curve* (or, more exactly, *the integral over a curve of the first type, or the integral over the length of an arc*). It is denoted

$$\int_L f(x, y)\, ds = \lim_{\substack{n\to\infty \\ \max \Delta_k s \to 0}} \sum_{k=1}^{n} f(\xi_k, \eta_k)\, \Delta_k s. \tag{5.117}$$

The curve L is called the *path* or *contour of integration.*

The definition of the integral along a curve $\int_L f(x, y, z)\, dx$ of a function of *three* variables (a space curve) does not require any modification.

In order to calculate an integral over a curve of the first type, it has to be reduced to an ordinary definite integral. Thus, if the space curve is given parametrically by means of the equations

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad t_0 \leqslant t \leqslant t_1,$$

then

$$\int_L f(x, y, z)\, ds$$

$$= \int\int_{t_0}^{t_1} f[x(t), y(t), z(t)] \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt. \tag{5.118}$$

The same formula, containing x and y only, can be used to calculate the integral over a curve in a plane.

The values of the function $f(x, y, z)$ can be given also on the surface S. Suppose that we subdivide the surface into elementary areas, whose size we denote by $\Delta_k\sigma$. Then, having selected a point (ξ_k, η_k, ζ_k) in each of these areas, we construct the integral sum

$$\sum_{k=1}^{m} f(\xi_k, \eta_k, \zeta_k)\,\Delta_k\sigma, \tag{5.119}$$

whose limit is called *the integral (of first type) over the surface S of the function $f(x, y, z)$*. It is denoted by the symbol

$$\int\int_S f(x, y, z)\,d\sigma \tag{5.120}$$

and it is calculated by means of a reduction to a double integral. If the surface S is given by means of an explicit equation $z = z(x, y)$ in the region G of the plane xOy, then

$$\int\int_S f(x, y, z)\,d\sigma$$
$$= \int\int_G f[x, y, z(x, y)]\sqrt{1 + (z'_x)^2 + (z'_y)^2}\,dx\,dy. \tag{5.121}$$

More general cases are considered in Chapter VII.

We now give a general definition of an integral over a manifold of a smaller number of dimensions. Suppose a function of n variables $f(x_1, x_2, \ldots, x_n)$ is defined in an m-dimensional manifold R $(m < n)$ possessing an m-dimensional volume (see section 3). Subdivide R into elementary parts, whose m-dimensional volumes we denote by $\Delta_k v$, and select a point $(\xi_1^{(k)}, \xi_2^{(k)}, \ldots, \xi_n^{(k)})$. Then it is possible to construct *the integral sum*

$$\sum f(\xi_1^{(k)}, \xi_2^{(k)}, \ldots, \xi_n^{(k)})\,\Delta_k v, \tag{5.122}$$

whose limit is called *the integral* of a function of n variables in an m-dimensional manifold *over the measure of the region.* The properties of integrals over the measure of the region are fully analogous to the properties of the ordinary definite, double, etc., integrals (see Chapter VII, § 2, sec. 1).

5. Alongside the integrals over the measure of the region it is also possible to define *integrals of the second type* in an m-dimensional manifold – *integrals in coordinates*. Integral sums, whose limit gives integrals of the second type, can be obtained from integral sums of section 4 by means of the exchange of the m-dimensional volume of an elementary part of the manifold R for an m-dimensional volume of a projection of this part into m-dimensional hyperplanes. Let us quote particular cases for $n = 2$ and $n = 3$.

Let $f(x, y)$ be a function of two variables defined along the plane curve L. Let us subdivide the curve L into elementary segments, each of which we then project onto the coordinate axes Ox and Oy. Then it is possible to construct two integral sums

$$\sum_{k=1}^{n} f(\xi_k, \eta_k)\, \Delta_k x \quad \text{and} \quad \sum_{k=1}^{n} f(\xi_k, \eta_k)\, \Delta_k y, \tag{5.123}$$

whose limits, if they exist, give *two integrals over a curve in coordinates (of the second type)*

$$\int_L f(x, y)\, dx \quad \text{and} \quad \int_L f(x, y)\, dy. \tag{5.124}$$

It is usual to consider, as an integral over a curve of the second type, a *composite* integral

$$\int_L P(x, y)\, dx + Q(x, y)\, dy, \tag{5.125}$$

where P and Q are arbitrary integrable functions.

For a space curve, the elementary segments of an arc can be projected onto the axes Ox, Oy and Oz, which gives three integral sums

$$\left.\begin{gathered} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k)\, \Delta_k x, \quad \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k)\, \Delta_k y, \\ \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k)\, \Delta_k z \end{gathered}\right\} \tag{5.126}$$

and, therefore, three integrals

$$\int_L f(x, y, z)\, dx, \quad \int_L f(x, y, z)\, dy, \quad \int_L f(x, y, z)\, dz. \tag{5.127}$$

As for a plane curve, often a composite integral is considered:

$$\int_L P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz. \tag{5.128}$$

The path of integration is assumed in both cases to be oriented. This means that a definite direction of motion along the curve is given, which is regarded as positive.

If the curve is an open one, an indication is given as to which of the two end-points is its beginning and which is the end. For a closed curve in the case of a plane, the chosen positive direction is usually anticlockwise, i.e. in tracing out the contour bounding the region the latter remains on the left.

If the function $f(x, y, z)$ is given on the surface S, then, instead of considering the integral $\int\int_S f(x, y, z)\, d\sigma$, we can consider the composite integral

$$\int\int_S P(x, y, z)\, dy\, dz + Q(x, y, z)\, dz\, dx + R(x, y, z)\, dx\, dy, \tag{5.129}$$

where P, Q, R are integrable functions of three variables. The integral $\int\int_S P\, dy\, dz$ is defined as the limit of the integral sum $\sum_{k=1}^{n} P(\xi_k, \eta_k, \zeta_k)\Delta_k\sigma(x, y)$, while $\Delta_k\sigma(xy)$ denotes the area of the projection of an elementary area onto the plane xOy. Two other integrals are defined analogously.

The surface S should be two-sided. This property is defined as follows. At every point of the surface, let us select a definite direction of the normal, which alters continuously while the point moves over the surface. Let the moving point describe an arbitrary closed curve, not intersecting the boundary of the surface. If, on the return of the point to its original position the direction of the normal in it coincides with the original one, the surface is called *two-sided* or *orientable.*

In order to construct an integral over the surface, the latter has to be orientable, i.e. a side to be regarded as positive has to be selected. The choice of a side is conditioned by the choice of one of two possible directions of a normal to the surface at every point,

which in turn influences the sign assigned to the value of the projection $\Delta_k\sigma$. This value is taken with the sign plus, if the normal forms an acute angle with the axis perpendicular to the corresponding coordinate plane, and with a minus in the opposite case.

For a closed surface, the side selected to be positive is usually the one at which the normal is directed outside the region bounded by that surface.

The properties and methods of calculation of integrals in coordinates and also their connection with integrals over the measure of the region are considered in Chapter VII. Here we demonstrate only the application of integrals over curves of the second type for the construction of functions of several variables from complete differentials.

6. Let $P(x)$ and $Q(x)$ be functions of two variables, continuous together with their partial derivatives in the *simply connected* closed squarable region G. The differential expression $Pdx + Qdy$ is called *integrable in the region G* if it represents a complete differential of a function of two variables, i.e. if there exists a function U, differentiable in the region G, such that

$$\frac{\partial U}{\partial x} = P(x, y), \quad \frac{\partial U}{\partial y} = Q(x, y). \tag{5.130}$$

THEOREM 10. *In order that the expression $Pdx + Qdy$ be integrable, it is necessary and sufficient that at every point of the region G the equation*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

should be satisfied.

This equation is called *the condition of integrability.*

The condition of integrability is also the necessary and sufficient condition that the integral $\int_L Pdx + Qdy$ is independent of the path Γ of integration provided the end-points of Γ are fixed. Thus we have the following theorem:

THEOREM 11. *The integral over a curve $\int_L Pdx + Qdx$ does not depend on the path of integration, if, and only if, the expression $Pdx + Qdx$ is a complete differential of a function of two variables.*

Since the integral does not depend on the path, it is sufficient to indicate, instead of the path of integration, the points (x_0, y_0) and (x, y) which serve as the ends of the arc of the curve L.

THEOREM 12. *The integral* $\int_{(x_0,y_0)}^{(x,y)} Pdx + Qdy$, *regarded as a function of the point* (x, y), *is the original function for the expression* $Pdx + Qdy$ *under the integral sign. If* $U(x, y)$ *is one of such originals, then*

$$\int_{(x_0,y_0)}^{(x,y)} P(x, y)\,dx + Q(x, y)\,dy = U(x, y) - U(x_0, y_0), \tag{5.131}$$

i.e. the integral over a curve of a complete differential equals the increment of the original function (the Newton–Leibniz formula).

An analogous property holds also for the integral of a space curve. The differential expression $Pdx + Qdy + Rdz$, where P, Q, R are functions of three variables, continuous together with their partial derivatives in the cubable *simply connected* region G, is called *integrable in* G, if it is a complete differential of a function of three variables.

In order that the expression $Pdx + Qdy + Rdz$ be integrable, it is necessary and sufficient that the conditions

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \tag{5.132}$$

are fulfilled at every point G *(conditions of integrability)*. The same conditions are necessary and sufficient for the independence of the integral over a curve $Pdx + Qdy + Rdz$ of the path of integration.

THEOREM 13. *The integral over a curve*

$$\int_L P\,dx + Q\,dy + R\,dz$$

does not depend on the path of integration if, and only if, the expression $Pdx + Qdy + Rdz$ *is a complete differential of a function of three variables. The integral*

$$\int_{(x_0,y_0,z_0)}^{(x,y,z)} P\,dx + Q\,dy + R\,dz,$$

regarded as a function of the point (x, y, z), is the original function for the expression $Pdx + Qdy + Rdz$ under the integral sign. If $U(x, y, z)$ is one of such originals, then

$$\int_{(x_0,y_0,z_0)}^{(x,y,z)} P\,dx + Q\,dy + R\,dz = U(x, y, z) - U(x_0, y_0, z_0), \tag{5.133}$$

i.e. the integral over a curve of a complete differential equals the increment of the original function (the Newton–Leibniz formula).

The integral over a curve of a function of n variables can be used in the same way for the reconstruction of a function of n variables from its complete differential.

§ 5. The Application of Definite Integrals to Problems of Geometry and Mechanics

1. The Evaluation of Areas of Plane Figures

1°. The area Q of a curvilinear trapezium bounded by the graph of a continuous, positive in the interval $[a, b]$, function $y = y(x)$ by

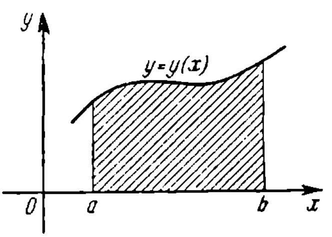

Fig. 15

the Ox axis and by two straight lines $x = a$ and $x = b$ (Fig. 15) equals

$$Q = \int_a^b y(x)\,dx. \tag{5.134}$$

If the function $y(x)$ is of alternating sign, the formula for the area should be applied separately to each segment, in which $y(x)$ preserves a constant sign, and add up the absolute values of the figures obtained.

2°. If the equations of a curve are given in a parametric form $x = x(t)$, $y = y(t)$, while $a = x(t_0)$, $b = x(T)$, then the area of the trapezium equals†

$$Q = \int_{t_0}^{T} y(t)\, x'(t)\, dt. \tag{5.135}$$

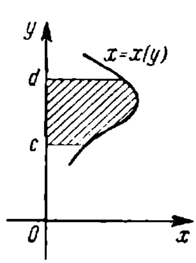

FIG. 16

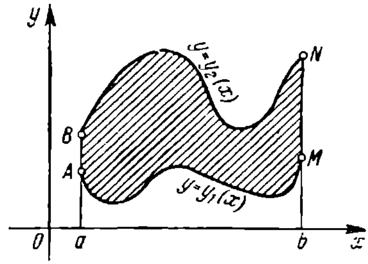

FIG. 17

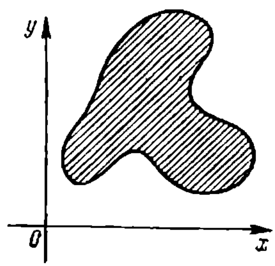

FIG. 18

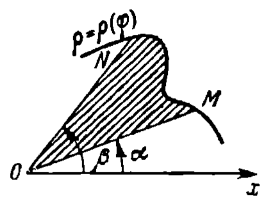

FIG. 19

3°. If the trapezium is bounded by the graph of the function $x = x(y)$, by the Oy axis and by the two straight lines $y = c$, $y = d$ (Fig. 16) then

$$Q = \int_{c}^{d} x(y)\, dy. \tag{5.136}$$

4°. The area Q of the figure $AMNB$ (Fig. 17) bounded by two continuous curves $y = y_1(x)$, $y = y_2(x)$ $(y_2(x) \geqslant y_1(x))$ and by two straight lines $x = a$ and $x = b$ equals

$$Q = \int_{a}^{b} [y_2(x) - y_1(x)]\, dx. \tag{5.137}$$

† Here and further, it is assumed that the function under the integral sign is positive and the upper limit of integration is greater than the lower one.

5°. If $x = x(t)$, $y = y(t)$ ($t_0 \leqslant t \leqslant T$) are parametric equations of a piecewise smooth, simple closed curve which bounds on its left a figure of area Q (Fig. 18), then

$$Q = -\int_{t_0}^{T} y(t)\, x'(t)\, dt = \int_{t_0}^{T} x(t)\, y'(t)\, dt$$

$$= \tfrac{1}{2}\int_{t_0}^{T} [x(t)\, y'(t) - y(t)\, x'(t)]\, dt. \tag{5.138}$$

6°. The area Q of the curvilinear sector OMN (Fig. 19), bounded by the continuous curve given by the equation in polar coordinates $\varrho = \varrho(\varphi)$ and by two rays $\varphi = \alpha$, $\varphi = \beta$ equals

$$Q = \tfrac{1}{2}\int_{\alpha}^{\beta} [\varrho(\varphi)]^2\, d\varphi. \tag{5.139}$$

7°. The area Q of a plane region D situated in the plane xOy equals

$$Q = \int\int_D dq = \int\int_D dx\, dy = \int\int_D \varrho\, d\varrho\, d\varphi. \tag{5.140}$$

8°. The area Q of a plane region D bounded by a simple piecewise smooth closed contour C equals

$$Q = -\oint_C y\, dx = \oint_C x\, dy = \tfrac{1}{2}\oint_C x\, dy - y\, dx. \tag{5.141}$$

2. The Calculation of the Lengths of Arcs of Curves

1°. The length s of the arc AB (Fig. 20) of a segment of a smooth (continuously differentiable) curve $y = y(x)$ ($a \leqslant x \leqslant b$) equals

$$s = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx, \tag{5.142}$$

and of the curve $x = x(y)$ ($c \leqslant y \leqslant d$) equals

$$s = \int_c^d \sqrt{1 + [x'(y)]^2}\, dy. \tag{5.143}$$

2°. If the curve is given by equations in the parametric form $x = x(t)$, $y = y(t)$ $(t_0 \leqslant t \leqslant T)$, where $x(t)$, $y(t)$ are continuously differentiable in the segment $[t_0, T]$ of the function, then the length of the arc of the curve s, equals

$$s = \int_{t_0}^{T} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt. \tag{5.144}$$

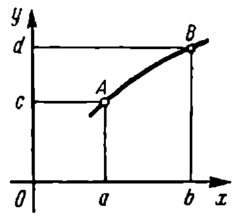

FIG. 20

3°. In polar coordinates the length s of the arc of the segment of the curve $\varrho = \varrho(\varphi)$ $(\alpha \leqslant \varphi \leqslant \beta)$, where $\varrho(\varphi)$ is a function, continuous together with its derivative $\varrho'(\varphi)$, equals

$$s = \int_{\alpha}^{\beta} \sqrt{[\varrho(\varphi)]^2 + [\varrho'(\varphi)]^2}\, d\varphi. \tag{5.145}$$

3. The Evaluation of Surface Areas

1°. The area Q of a smooth curved surface $z = z(x, y)$ equals

$$Q = \iint_D \frac{dq}{\cos\gamma} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy, \tag{5.146}$$

where D is the projection of a surface onto the plane xOy and γ is the angle formed by the normal to the element of surface and the axis Oz (the angle between the element of surface and the plane xOy).

2°. If the surface is given by parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, where the point $(u, v) \in D$ and D is a bounded closed squarable region in which the functions x, y, z are continuously differentiable, then the area Q of the surface equals

$$Q = \iint_D \sqrt{EG - F^2}\, du\, dv, \tag{5.147}$$

where

$$\left.\begin{aligned} E &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \\ G &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2, \\ F &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}. \end{aligned}\right\} \quad (5.148)$$

3°. If the surface is given by an equation in cylindrical coordinates $z = z(\varrho, \varphi)$, then the area Q equals

$$Q = \iint_D \sqrt{\varrho^2 + \varrho^2\left(\frac{\partial z}{\partial \varrho}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2}\, d\varrho\, d\varphi. \quad (5.149)$$

4°. The area Q of a surface formed by the revolving of the arc $\overset{\smile}{AB}$ of a smooth plane curve $y = y(x)$ $(a \leqslant x \leqslant b)$ about the axis Ox (Fig. 21) equals

$$Q = 2\pi \int_{(A)}^{(B)} y\, ds = 2\pi \int_a^b y(x)\sqrt{1 + [y'(x)]^2}\, dx. \quad (5.150)$$

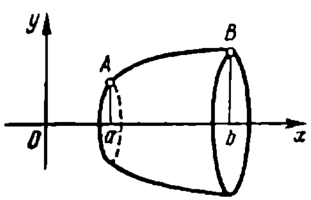

FIG. 21

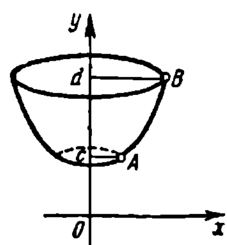

FIG. 22

5°. If a plane curve is given by equations in a parametric form $x = x(t)$, $y = y(t)$ $(t_0 \leqslant t \leqslant T)$ then the area Q of the surface of revolution (about the axis Ox) equals

$$Q = 2\pi \int_{(A)}^{(B)} y\, ds = 2\pi \int_{t_0}^{T} y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt. \quad (5.151)$$

6^0. The area Q of a surface formed by the revolution of an arc $\breve{AB}$ of a smooth plane curve $x = x(y)$ $(c \leqslant y \leqslant d)$ about the Oy axis (Fig. 22) equals

$$Q = 2\pi \int_{(A)}^{(B)} x\, ds = 2\pi \int_c^d x(y) \sqrt{1 + [x'(y)]^2}\, dy. \quad (5.152)$$

4. The Evaluation of Volumes

1^0. The volume v of a cylindroid, bounded from above by the continuous surface $z = z(x, y)$ or, in cylindrical coordinates $z = z(\varrho, \varphi)$ and from below by the plane $z = 0$ and on the sides by

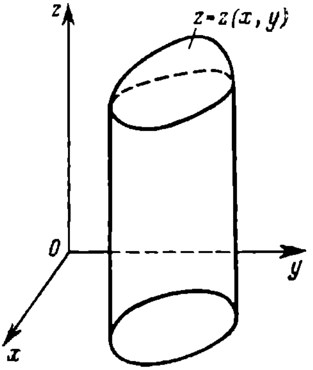

Fig. 23

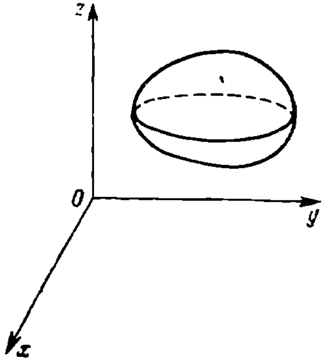

Fig. 24

a right cylindrical surface, whose intersection with the plane xOy is the squarable region D (Fig. 23) equals

$$v = \iint_D z\, dq = \iint_D z\, dx\, dy = \iint_D z\varrho\, d\varrho\, d\varphi. \quad (5.153)$$

2^0. The volume v of a closed space region V (Fig. 24) equals

$$v = \iiint_V dv. \quad (5.154)$$

Here, if the surfaces bounding the surface are given in cartesian coordinates, then

$$v = \iiint_V dx\, dy\, dz, \quad (5.155)$$

if they are given in cylindrical coordinates then

$$v = \iiint_V \varrho \, dz \, d\varrho \, d\varphi, \tag{5.156}$$

and if they are given in spherical coordinates then

$$v = \iiint_V \varrho^2 \sin \vartheta \, d\varrho \, d\varphi \, d\vartheta. \tag{5.157}$$

3°. The volume v of a body of arbitrary form, enclosed between two planes $x = a$ and $x = b$, perpendicular to the axis Ox, and

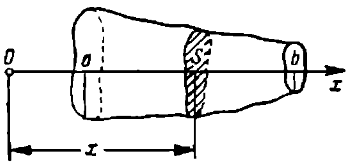

FIG. 25

with an area of cross-section $S = Q(x)$ when the body is traversed by a plane perpendicular to the Ox axis at the point x (Fig. 25) equals

$$v = \int_a^b Q(x) \, dx. \tag{5.158}$$

4°. The volume v of a body formed by the revolution about the Ox axis of a plane figure, bounded by the curve $y = y(x)$, where $y(x)$ is a continuous non-negative function, by the Ox axis and by two straight lines $x = a$ and $x = b$, equals

$$v = \pi \int_a^b [y(x)]^2 \, dx. \tag{5.159}$$

5°. In a more general case, the volume v of a ring formed by the revolution about the Ox axis of a plane figure bounded by two curves $y = y_1(x)$, $y = y_2(x)$, where $y_1(x)$ and $y_2(x)$ are continuous non-negative functions, and by two straight lines $x = a$ and $x = b$, equals

$$v = \pi \int_a^b \{[y_2(x)]^2 - [y_1(x)]^2\} \, dx. \tag{5.160}$$

6^0. Similarly, the volume v of the solid of revolution formed by the revolution about the Oy axis of a plane figure, bounded by the curve $x = x(y)$, where $x(y)$ is a continuous non-negative function, by the axis Oy and by two straight lines $y = c$ and $y = d$, equals

$$v = \pi \int_c^d [x(y)]^2 \, dy. \tag{5.160a}$$

5. Guldin's Theorem

Theorem 14 *(Guldin's first theorem). If an arc of a plane curve of length L revolves about an axis, with which it does not intersect, and the curve lies in the same plane as the axis, the area of the surface of the solid so formed is calculated from the formula*

$$Q = L \times 2\pi d, \tag{5.161}$$

where d is the distance of the centre of gravity of the arc from the axis of rotation.

Theorem 15 *(Guldin's second theorem). If a plane figure of area Q revolves about an axis, with which it does not intersect and which lies in the same plane as the curve, the volume of the solid so formed is calculated from the formula*

$$v = Q \times 2\pi d, \tag{5.162}$$

where d is the distance of the centre of gravity of the area of the figure from the axis of rotation.

6. The Calculation of Mass

1^0. The calculation of mass M of a curvilinear segment l with a variable linear density δ is carried out according to the formula

$$M = \int_l \delta \, ds, \tag{5.163}$$

where $\delta = \delta(x, y)$ for a plane curve and $\delta = \delta(x, y, z)$ for a space curve.

2^0. The mass M of a plane figure D with a variable surface density δ equals

$$M = \iint_D \delta \, dq. \tag{5.164}$$

In cartesian coordinates this formula acquires the form

$$M = \iint_D \delta(x, y)\, dx\, dy, \tag{5.165}$$

and in polar coordinates

$$M = \iint_D \delta(\varrho, \varphi)\, \varrho\, d\varrho\, d\varphi. \tag{5.166}$$

3°. The mass M of a curvilinear figure S with a variable surface density $\delta = \delta(x, y, z)$ equals

$$M = \iint_S \delta(x, y, z)\, d\sigma. \tag{5.167}$$

4°. The mass M of a body V with a variable solid density δ equals

$$M = \iiint_V \delta\, dv. \tag{5.168}$$

In cartesian coordinates this formula takes the form

$$M = \iiint_V \delta(x, y, z)\, dx\, dy\, dz, \tag{5.169}$$

in cylindrical coordinates

$$M = \iiint_V \delta(\varrho, \varphi, z) \varrho\, d\varrho\, d\varphi\, dz, \tag{5.170}$$

and in spherical coordinates

$$M = \iiint_V \delta(\varrho, \varphi, \theta)\, \varrho^2 \sin\theta\, d\varrho\, d\varphi\, d\theta. \tag{5.171}$$

7. The Calculation of the Coordinates of a Centre of Gravity

1°. The coordinates of the centre of gravity C of an arc of a homogeneous plane curve $y = y(x)$ $(a \leqslant x \leqslant b)$ of length L are determined from the formulae

$$x_C = \frac{\int_a^b x\sqrt{1 + [y'(x)]^2}\, dx}{L}, \quad y_C = \frac{\int_a^b y(x)\sqrt{1 + [y'(x)]^2}\, dx}{L}. \tag{5.172}$$

For a closed curve

$$\left.\begin{aligned} x_C &= \frac{\displaystyle\int_a^b x\left\{\sqrt{1+[y_1'(x)]^2}+\sqrt{1+[y_2'(x)]^2}\right\}dx}{L}, \\ y_C &= \frac{\displaystyle\int_a^b \left\{y_1(x)\sqrt{1+[y_1'(x)]^2}+y_2(x)\sqrt{1+[y_2'(x)]^2}\right\}dx}{L}, \end{aligned}\right\} \tag{5.173}$$

where $y = y_1(x)$ and $y = y_2(x)$ are equations of the upper and lower parts of the contour respectively, and L is the length of the entire contour.

2°. The coordinates of the centre of gravity C of an arc of the curve l with a variable density δ and mass M equal

$$x_C = \frac{\displaystyle\int_l \delta x\,ds}{M}, \quad y_C = \frac{\displaystyle\int_l \delta y\,ds}{M}, \tag{5.174}$$

for a space curve

$$x_C = \frac{\displaystyle\int_l \delta x\,ds}{M}, \quad y_C = \frac{\displaystyle\int_l \delta y\,ds}{M}, \quad z_C = \frac{\displaystyle\int_l \delta z\,ds}{M}, \tag{5.175}$$

where $\delta = \delta(x, y, z)$.

3°. The coordinates of the centre of gravity C of a plane lamina D with variable density δ and mass $M = \displaystyle\iint_D \delta\,dq$ equal:

in cartesian coordinates

$$x_C = \frac{\displaystyle\iint_D \delta x\,dx\,dy}{\displaystyle\iint_D \delta\,dx\,dy}, \quad y = \frac{\displaystyle\iint_D \delta y\,dx\,dy}{\displaystyle\iint_D \delta\,dx\,dy}, \tag{5.176}$$

where $\delta = \delta(x, y)$;

in polar coordinates

$$x_C = \frac{\iint_D \delta\varrho^2 \cos\varphi \, d\varrho \, d\varphi}{\iint_D \delta\varrho \, d\varrho \, d\varphi}, \quad y_C = \frac{\iint_D \delta\varrho^2 \sin\varphi \, d\varrho \, d\varphi}{\iint_D \delta\varrho \, d\varrho \, d\varphi}, \tag{5.177}$$

where $\delta = \delta(\varrho, \varphi)$.

If the lamina is homogeneous, $\delta = 1$ in these formulae.

4°. The coordinates of the centre of gravity C of a body V with variable density δ and mass $M = \iiint_V \delta \, dv$ equal

$$x_C = \frac{\iiint_V \delta x \, dv}{M}, \quad y_C = \frac{\iiint_V \delta y \, dv}{M}, \quad z_C = \frac{\iiint_V \delta z \, dv}{M}; \tag{5.178}$$

in particular, in cartesian coordinates

$$\left.\begin{aligned} x_C &= \frac{\iiint_V \delta x \, dx \, dy \, dz}{\iiint_V \delta \, dx \, dy \, dz}, \quad y_C = \frac{\iiint_V \delta y \, dx \, dy \, dz}{\iiint_V \delta \, dx \, dy \, dz}, \\ z_C &= \frac{\iiint_V \delta z \, dx \, dy \, dz}{\iiint_V \delta \, dx \, dy \, dz}, \end{aligned}\right\} \tag{5.179}$$

where $\delta = \delta(x, y, z)$.

If the body is homogeneous, these formulae should have $\delta = 1$.

8. The Calculation of Moments of Inertia

1°. The moments of inertia I_x and I_y of a plane lamina D with variable density $\delta = \delta(x, y)$ with respect to the coordinate axes Ox and Oy are determined by the formulae

$$I_x = \iint_D \delta y^2 \, dx \, dy, \quad I_y = \iint_D \delta x^2 \, dx \, dy, \tag{5.180}$$

and the moment of inertia I_0 with respect to the origin O equals

$$I_0 = \iint_D \delta(x^2 + y^2)\, dx\, dy. \tag{5.181}$$

2°. The moment of inertia I_x of a plane lamina D of variable density $\delta = \delta(\varrho, \varphi)$, with respect to the polar axis Ox, equals

$$I_x = \iint_D \delta \varrho^3 \sin^2 \varphi\, d\varrho\, d\varphi, \tag{5.182}$$

and the moment I_0 is given by the equation

$$I_0 = \iint_D \delta \varrho^3\, d\varrho\, d\varphi. \tag{5.183}$$

3°. If I_l, I_{l_0} are moments of inertia of a plane figure D of area Q with respect to two parallel axes l and l_0, of which l_0 passes through the centre of gravity of the figure, and d is the distance between those axes, then

$$I_l = I_{l_0} + Qd^2. \tag{5.184}$$

4°. The moment of inertia I of a plane figure D with respect to a straight line passing through the centre of gravity $O(0, 0)$ and forming an angle α with the axis Ox, equals

$$I = I_x \cos^2 \alpha - 2I_{xy} \sin \alpha \cos \alpha + I_y \sin^2 \alpha, \tag{5.185}$$

where I_x and I_y are moments of inertia of the figure with respect to the axes Ox and Oy and I_{xy} is the centrifugal moment equal to

$$I_{xy} = \iint_D \delta xy\, dx\, dy. \tag{5.186}$$

5°. Moments of inertia I_{xy}, I_{yz}, I_{zx} of a body V with a variable density $\delta = \delta(x, y, z)$ with respect to coordinate planes are determined by the formulae

$$\left.\begin{aligned} I_{xy} &= \iiint_V \delta z^2\, dx\, dy\, dz, \\ I_{yz} &= \iiint_V \delta x^2\, dx\, dy\, dz, \\ I_{zx} &= \iiint_V \delta y^2\, dx\, dy\, dz. \end{aligned}\right\} \tag{5.187}$$

The moment of inertia of the body V with a variable density δ, with respect to the axis l, equals

$$I_l = \iiint_V \delta r^2\, dx\, dy\, dz, \tag{5.188}$$

where r is the distance of the variable point (x, y, z) from the axis l.

In particular, for coordinate axes we have

$$I_x = I_{xy} + I_{xz}, \quad I_y = I_{yx} + I_{yz}, \quad I_z = I_{zx} + I_{zy}; \tag{5.189}$$

for example in cartesian coordinates

$$I_z = \iiint_V \delta(x^2 + y^2)\, dx\, dy\, dz, \quad \text{where} \quad \delta = \delta(x, y, z), \tag{5.190}$$

in cylindrical coordinates

$$I_z = \iiint_V \delta\varrho^3\, d\varrho\, d\varphi\, dz, \quad \text{where} \quad \delta = \delta(\varrho, \varphi, z), \tag{5.191}$$

and in spherical coordinates

$$I_z = \iiint_V \delta\varrho^4 \sin^3\theta\, d\varrho\, d\varphi\, d\theta, \quad \text{where} \quad \delta = \delta(\varrho, \varphi, \theta). \tag{5.192}$$

6^0. The moment of inertia I_0 of a body V with variable density δ, with respect to the pole O, equals

$$I_0 = \iiint_V \delta r^2\, dv. \tag{5.193}$$

In particular, in cartesian coordinates

$$I_0 = \iiint_V \delta(x^2 + y^2 + z^2)\, dx\, dy\, dz, \quad \text{where} \quad \delta = \delta(x, y, z), \tag{5.194}$$

in cylindrical coordinates

$$I_0 = \iiint_V \delta(\varrho^2 + z^2)\varrho\, d\varrho\, d\varphi\, dz, \quad \text{where} \quad \delta = \delta(\varrho, \varphi, z), \tag{5.195}$$

and in spherical coordinates

$$I_0 = \iiint_V \delta\varrho^4 \sin\theta \, d\varrho \, d\varphi \, d\theta, \quad \text{where} \quad \delta = \delta(\varrho, \varphi, \theta). \tag{5.196}$$

7°. There exists a relationship

$$I_0 = I_{xy} + I_{yz} + I_{zx}. \tag{5.197}$$

8°. If I_l is the moment of inertia of a body with respect to some axis l, I_{l_0} is the moment of inertia with respect to an axis parallel to l and passing through the centre of gravity O of the body, d is the distance between the axes and M is the mass of the body, then

$$I_l = I_{l_0} + Md^2. \tag{5.198}$$

9°. The moment of inertia of a body V with respect to an axis l passing through its centre of gravity $O(0, 0, 0)$ and forming angles α, β, γ with the coordinate axes equals

$$\begin{aligned} I_l = I_x \cos^2\alpha + I_y \cos^2\beta + I_z \cos^2\gamma - 2K_{xy} \cos\alpha \cos\beta \\ - 2K_{xz} \cos\alpha \cos\gamma - 2K_{yz} \cos\beta \cos\gamma, \end{aligned} \tag{5.199}$$

where I_x, I_y and I_z are moments of inertia of the body with respect to the coordinate axes, and

$$\left.\begin{aligned} K_{xy} &= \iiint_V \delta xy \, dx \, dy \, dz, \\ K_{xz} &= \iiint_V \delta xz \, dx \, dy \, dz, \\ K_{yz} &= \iiint_V \delta yz \, dx \, dy \, dz \end{aligned}\right\} \tag{5.200}$$

are centrifugal moments.

9. The Calculation of the Potential of a Field of Gravity

The potential $u(x, y, z)$ of a field of gravity, or the Newton potential of a body V at the point $P(x, y, z)$ is the name given to the integral

$$u(x, y, z) = \iiint_V \delta(\xi, \eta, \zeta) \frac{d\xi \, d\eta \, d\zeta}{r}, \tag{5.201}$$

where $\delta = \delta(\xi, \eta, \zeta)$ is the density of the body and

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}. \qquad (5.202)$$

A material point of mass m is attracted by the body with a force $\mathbf{F}$, whose projections X, Y, Z onto the coordinate axes Ox, Oy and Oz equal

$$\left.\begin{aligned} X &= km \frac{\partial u}{\partial x} = km \int\int\int_V \delta \frac{\xi - x}{r^3} \, d\xi \, d\eta \, d\zeta, \\ Y &= km \frac{\partial u}{\partial y} = km \int\int\int_V \delta \frac{\eta - y}{r^3} \, d\xi \, d\eta \, d\zeta, \\ Z &= km \frac{\partial u}{\partial z} = km \int\int\int_V \delta \frac{\zeta - z}{r^3} \, d\xi \, d\eta \, d\zeta, \end{aligned}\right\} \qquad (5.203)$$

where k is the gravitational constant.

10. The Calculation of Work, Path and Force of Pressure

1°. The work A of a force $\mathbf{F} = \mathbf{F}(x)$ in displacing a material point along the straight line Ox from $x = a$ to $x = b$ $(a < b)$, provided the direction of the force is the same as that of the axis Ox, equals

$$A = \int_a^b F(x) \, dx. \qquad (5.204)$$

The work A which is done by the force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ in a space field of force in displacing a material point along a curve l equals

$$A = \int_l \mathbf{F}(\mathbf{r}) \, d\mathbf{r} = \int_l X \, dx + Y \, dy + Z \, dz, \qquad (5.205)$$

where $X(x, y, z)$, $Y(x, y, z)$, $Z(x, y, z)$ are projections of the force $\mathbf{F}$ onto the coordinate axes.

In a plane field of force

$$A = \int_l X \, dx + Y \, dy, \qquad (5.206)$$

where $X(x, y)$ and $Y(x, y)$ are projections of the force $\mathbf{F}$ onto the coordinate axes.

2°. The path s traversed by a point moving in a straight line with velocity $v = v(t)$ in the time-interval from $t = t$ to $t = T$ equals

$$s = \int_{t_0}^{T} v(t)\, dt. \tag{5.207}$$

3°. The pressure p exerted by a liquid of specific gravity γ on one side of a vertical lamina immersed in it, when the distance x of the points of the lamina from the level of the liquid varies from $x = a$ to $x = b$ and the length y of a horizontal section through the lamina is a function of $x : y = y(x)$, equals

$$p = \int_{a}^{b} \gamma x y(x)\, dx. \tag{5.208}$$

CHAPTER VI

IMPROPER INTEGRALS. INTEGRALS DEPENDENT ON A PARAMETER. STIELTJES' INTEGRAL

§ 1. Improper Integrals

1. In the definition of the integral $\int_a^b f(x)\,dx$, given in Chapter V, § 3, it was assumed that the interval of integration $[a, b]$ is *finite* and the function $f(x)$ is continuous in this interval, or has a *finite* number of discontinuities of the first kind. The extension of the concept of a definite integral to cover the case when the function $f(x)$ has an *arbitrary set* of points of discontinuity, while remaining *bounded*, leads to Riemann's integral (Chapter V, § 3) and to its further generalizations, studied in the theory of real variables.

In this chapter, we investigate cases when either the interval of integration becomes *infinite*, or the integrand ceases being bounded; here it is unbounded in the neighbourhood of a finite number of points only, points which can be situated at the ends of the interval as well as inside it. In all these cases, in order to define the integral, it is necessary to carry out one more limiting process. Integrals, obtained as a result of this new limiting process are called *improper*; the integrals in ordinary sense are henceforth called *proper* ones.

2. Suppose that the function $f(x)$ is defined in the semi-open interval $[a, +\infty)$ and is properly integrable in the interval $[a, l]$, where l is any number greater than a.

The name *improper integral* of the function $f(x)$ in the semi-open interval $[a, +\infty)$ is applied to the limit of the proper integral taken in the interval $[a, l]$ on condition that l tends to infinity, i.e.

$$\int_a^{\infty} f(x)\,dx = \lim_{l\to\infty}\int_a^{l} f(x)\,dx, \tag{6.1}$$

where the symbol $\int_a^\infty f(x)\,dx$ is the notation of an improper integral.

If this limit exists, the improper integral is called *convergent*. In the contrary case, the integral *does not exist* or *diverges*. In this case, two possibilities should be distinguished:

(1) $\lim\limits_{l\to\infty} \int_a^l f(x)\,dx$ equals $+\infty$ or $-\infty$,

(2) $\int_a^l f(x)\,dx$ does not tend to a limit at all (whether finite or infinite) i.e. it is an oscillating function of its upper limit l.

If the original function $F(x)$ of the function $f(x)$ is known, it is easy to verify directly whether the improper integral is convergent or not. Since

$$\int_a^\infty f(x)\,dx = \lim_{l\to\infty} [F(l) - F(a)] = F(\infty) - F(a), \tag{6.2}$$

the question of convergence of the integral is reduced to revealing the existence of a limit of the function $F(x)$ when $x \to \infty$.

EXAMPLE 1.

$$\int_0^\infty e^{-x}\,dx = \lim_{l\to\infty} (-e^{-l} + 1) = 1,$$

therefore $\int_0^\infty e^{-x}\,dx$ is convergent.

EXAMPLE 2.

$$\int_1^\infty \frac{dx}{x} \lim_{l\to\infty} \ln l = \infty.$$

EXAMPLE 3.

$$\int_0^\infty \sin x\,dx = \lim_{l\to\infty} (1 - \cos l) \text{ does not exist.}$$

In the second and third examples the improper integral diverges, and in the second example it tends to infinity and in the third example it oscillates between 0 and 2.

Geometrically, the improper integral of the type under consideration, on converging, expresses the area between a curve and its asymptote.

The improper integral in the semi-open interval $(-\infty, a]$ is defined in an analogous way:

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{l\to-\infty}\int_{l}^{a} f(x)\,dx. \tag{6.3}$$

If both limits of integration are infinite (here it is assumed, that the function $f(x)$ is defined over the entire numerical axis and is properly integrable in any interval), then, by definition,

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx, \tag{6.4}$$

where a is any number, on whose choice the improper integral does not depend. The latter definition can also be written down in the form

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{l\to-\infty,\\ l'\to+\infty}}\int_{l}^{l'} f(x)\,dx; \tag{6.5}$$

here the limiting processes $l \to -\infty$ and $l' \to +\infty$ are carried out *independently* of each other. If the original function $f(x)$ is known and its limits for $x \to +\infty$ and $x \to -\infty$ exist, then

$$\int_{-\infty}^{+\infty} f(x)\,dx = F(+\infty) - F(-\infty). \tag{6.6}$$

If a least one of the limits shown does not exist, the integra diverges.

EXAMPLE 4.

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \arctan(+\infty) - \arctan(-\infty) = \pi.$$

In the examples quoted, the integrand tended to zero, when $x \to \infty$, for convergent integrals. However, this is *by no means a necessary condition* of convergence of an integral. Geometrically, it is clear that if an improper integral converges and at the same time the integrand tends to some limit, this limit can only be zero (the fact that the tendency of the integrand to zero is not a sufficient condition of convergence of the integral is shown by Example 2).

However, an integral may converge even when the integrand does not end to any limit. An example of that is afforded by diffraction integrals *(Fresnel integrals)*

$$\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Moreover, it can be shown, for example, that the integral

$$\int_0^\infty \frac{x}{1 + x\sin^2 x}\, dx$$

converges in spite of the fact that the integrand, remaining positive all the time, is not even bounded ($f(k\pi) = k\pi$). The graph of this function has an infinite number of "spikes", whose height increases unlimitedly, and the width of the "base" tends to zero. At points

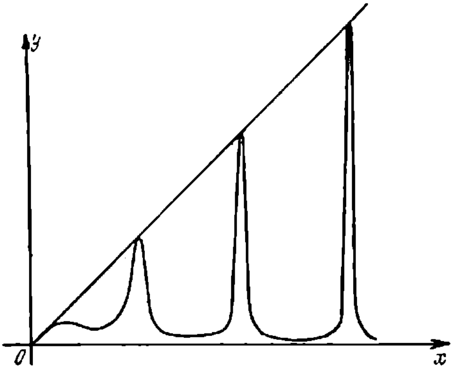

FIG. 26

lying outside the bases of these spikes, the function tends swiftly to zero, when $x \to \infty$ (Fig. 26).

In the cases when the original function $F(x)$ is unknown, the investigation for convergence of an integral can create considerable difficulties. Usually, in such an investigation use is made of tests for convergence (or existence) of an improper integral.

3. In the following only the improper integral $\int_0^\infty f(x)\, dx$ is considered, since the tests of convergence in the two remaining cases are formulated completely analogously.

The convergence of the integral $\int_a^\infty f(x)\,dx$ is equivalent to the convergence of the integral $\int_N^\infty f(x)\,dx$, where N is any number greater than a, since the difference of these integrals is a proper integral.

CAUCHY TEST FOR CONVERGENCE. For the convergence of the integral $\int_a^\infty f(x)\,dx$ it is necessary and sufficient that to every number, positive and as small as desired, there can be found a corresponding number l, such that for any $q > p > l$ the following inequality holds:

$$\left|\int_p^q f(x)\,dx\right| < \varepsilon. \tag{6.7}$$

This test is a direct consequence of Cauchy's general test for convergence of numerical sequences.

In general, it should be noted that there is much in common between the tests of convergece of improper integrals and the tests of convergence of numerical series (or sequences). The convergence of an improper integral with a positive integrand is equivalent to the convergence of the series $\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)\,dx$, where $a_0 = a$ and the sequence $\{a_n\}$ increasing monotonically tends to infinity (on other counts, the selection of the sequence $\{a_n\}$ is arbitrary). This follows from the monotonic increase of the integral $\int_a^l f(x)\,dx$ with the increase of l, in consequence of which it is possible to find, for any number l, a k, that

$$s_k \leqslant \int_a^l f(x)\,dx \leqslant s_{k+1},$$

where s_k is a partial sum of the series. Here, the condition that the integrand be positive is of importance. This is shown by the example of the integral $\int_0^\infty \sin x\,dx$, which is divergent, while

$$\sum_{n=0}^{\infty} \int_{2n\pi}^{2(n+1)\pi} \sin x\,dx = 0.$$

In this example it is easy to detect an analogy with the divergent series $1 - 1 + - 1 + \cdots$ which becomes convergent if each two neighbouring terms are joined together.

Just as in the study of numerical series, the idea of *absolute convergence* of improper integrals plays an important role. An improper integral is called *absolutely convergent* if it converges after

substituting for the integrand $f(x)$ its absolute value, i.e. if $\int_0^\infty |f(x)|\,dx$ is convergent.

Any absolutely convergent integral converges. This follows from the inequality

$$\left|\int_p^q f(x)\,dx\right| \leqslant \int_p^q |f(x)|\,dx$$

and the Cauchy test for convergence.

An integral which converges but does not converge absolutely is said to *converge conditionally.*

The signs of absolute convergence of an integral are, as a rule, based on the fact that if $\int_a^l |f(x)|\,dx$ remains bounded for $l \to \infty$, then, on the basis of Weierstrass' test for convergence, it converges.

COMPARISON TESTS. 1°. *Suppose that for all values $x > N \geqslant a$ the inequality*

$$|f(x)| \leqslant |\varphi(x)|$$

holds.

If the second of the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty \varphi(x)\,dx$ converges absolutely, then the first also does. If the first integral does not converge absolutely, then neither can the second one.

2°. *If the function $\varphi(x)$ does not alter in sign for sufficiently large x and*

$$\lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = A \neq 0, \tag{6.8}$$

then either both integrals are absolutely convergent, or both are absolutely divergent.

The test 2° is found to be more convenient in practice than the test 1°. The functions most frequently used as a comparison function is $1/x^\alpha$ $(\alpha > 0)$. For this function $\int_a^\infty dx/x^\alpha$ $(a > 0)$ converges when $\alpha > 1$ and diverges when $\alpha \leqslant 1$.

If the function $f(x)$ has the form $\psi(x)/x^\alpha$, where $\psi(x)$ is a function bounded when $x \to \infty$, the integral $\int_a^\infty f(x)\,dx$ $(a > 0)$ is absolutely convergent for $\alpha > 1$. If, on the other hand, $\alpha \leqslant 1$ and the function $\psi(x)$ preserves a constant sign and does not tend to zero, the given integral diverges.

Using the test given above, it is possible to establish, for example, the convergence of the integrals $\int_1^\infty \frac{dx}{x\sqrt{1+x^3}}$, $\int_0^\infty xe^{-\alpha x} dx$, when $n > 0$, $\alpha > 0$. Applying this test, it is possible to establish that integrals $\int_0^\infty \frac{\sin x}{1+x^2} dx$ and $\int_0^\infty \frac{\cos x \, dx}{x(1+\sqrt{x})}$ are absolutely convergent, etc. On being applied to the integral $\int_1^\infty \frac{\sin x}{x} dx$ the test does not give an answer, since, although $\alpha = 1$, the function $\sin x$ is of alternating sign.

It is particularly simple to apply the given test if $f(x)$ is a rational function; in this case, it is sufficient and necessary for the convergence of the integral, that the power of the denominator should be *greater by at least two* than the power of the numerator.

TESTS OF NON-ABSOLUTE CONVERGENCE OF AN INTEGRAL. 1^0. *Suppose the functions $f(x)$ and $g(x)$ are defined in the semi-open interval $[a, \infty)$, and $\int_a^\infty f(x)\,dx$ converges and $g(x)$ is monotonic and bounded. Then $\int_a^\infty f(x)\,g(x)\,dx$ converges.*

Indeed, from the second theorem about mean value (see Chapter V, § 3)

$$\int_p^q f(x)\,g(x)\,dx = g(p)\int_p^\xi f(x)\,dx + g(q)\int_\xi^q f(x)\,dx,$$

where $p < \xi < q$. The given test follows from the Cauchy test applied to the convergent integral $\int_a^\infty f(x)\,dx$.

2^0. *If the function $f(x)$ has a proper integral in any interval $[a, l]$ $(l > a)$ and the integral $\left|\int_a^1 f(x)\,dx\right| < M$, where M is a constant, and the function $g(x)$ tends monotonically to zero for $x \to \infty$, the integral $\int_a^\infty f(x)\,g(x)\,dx$ is convergent.*

Both the above signs are analogous to the Abel and Dirichlet tests of convergence of numerical series. On the basis of test 2^0, the following integrals, for example, are convergent:

$$\int_0^\infty \frac{\sin x}{x^\alpha} dx \quad \text{and} \int_0^\infty \frac{\cos x}{x^\alpha} dx \quad \text{for} \quad \alpha > 0.$$

In particular, for $\alpha = 1$ we obtain the convergence of the integral $\int_0^\infty \frac{\sin x}{x}$ *(Dirichlet's integral)*, which equals

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{6.9}$$

Prove that this integral converges conditionally, i.e. that $\int_0^\infty \frac{|\sin x|}{x}\, dx$ diverges. We utilize the note made above and we consider the series

$$\sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\, dx.$$

Estimating an arbitrary term of this series

$$\int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\, dx > \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x|\, dx = \frac{2}{(n+1)\pi},$$

we find that it is greater than a term of the divergent series $\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+1}$.

In investigating integrals containing periodic functions, the following test is often useful.

3°. Suppose the function $f(x)$ is defined in the semi-open interval $[a, \infty)$ and has a period $\omega > 0$, and the function $g(x)$ is monotonic and tends to zero when $x \to \infty$.

If $\int_a^{a+\omega} f(x)\, dx = 0$, then $\int_0^\infty f(x)\, g(x)\, dx$ converges. If $\int_a^{a+\omega} f(x) \times dx \neq 0$, then $\int_0^\infty f(x) g(x) dx$ converges or diverges at the same time as the integral $\int_0^\infty g(x)\, dx$.

Additional notes. In the case when the corresponding original function is not expressible in terms of elementary ones, many improper integrals may be evaluated with the aid of special devices (among the more important of those are integration and differentiation with respect to a parameter (see § 2, secs. 3 and 4) and the application of the theory of deduction).

Certain improper integrals can be reduced to *Frullani integrals*,

$$I = \int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx \quad (a > 0, b > 0), \tag{6.10}$$

which are easily calculated given the following assumptions about the function $f(x)$.

1°. If $f(x)$ is continuous for $x \geqslant 0$, and there exists a finite limit $\lim_{x=\infty} f(x) = f(\infty)$, then

$$I = [f(0) - f(\infty)] \ln \frac{b}{a}. \tag{6.11}$$

2°. If $f(x)$ is continuous for $x \geqslant 0$ and the integral $\int_A^\infty [f(x)/x]\, dx$ converges for any $A > 0$, then

$$I = f(0) \ln \frac{b}{a}. \tag{6.12}$$

Integrals of Unbounded Functions

4. In investigating the integrals of unbounded functions it is sufficient to confine oneself to the case when there is only one point of infinite discontinuity in the interval $[a, b]$; in the contrary case, the interval $[a, b]$ should be subdivided into parts each of which contains only one point of the above type.

Let the function $f(x)$ be defined and continuous in the interval $[a, b]$ except at one point in it, where $(a \leqslant c \leqslant b)$, in which it increases unboundedly. The point c we shall call *a singular* one.

An improper integral of the function $f(x)$ in the interval $[a, b]$ is the name given to the sum of limits of proper integrals taken over the intervals $[a, c - \delta']$ and $[c + \delta'', b]$ on condition that δ' and δ'' tend to zero independently:

$$\int_a^b f(x)\, dx = \lim_{\delta' \to 0} \int_a^{c-\delta'} f(x)\, dx + \lim_{\delta'' \to 0} \int_{c+\delta''}^b f(x)\, dx. \tag{6.13}$$

If the point c coincides with the end a of the interval, there remains only one limit, and if with the end b then the first one.

If both the given limits exist, the improper integral is called *convergent*; otherwise, it is called *divergent*. In agreement with this definition $\int_a^b dx/(x - c)^\alpha$ is convergent and equals

$$\frac{1}{1 - \alpha} \left[\frac{1}{(b - c)^{\alpha - 1}} - \frac{1}{(a - c)^\alpha} \right]$$

if $\alpha < 1$, and diverges if $\alpha \geqslant 1$.

Just as before, the question of the convergence of the integral can be solved easily, if the original function $F(x)$ is known. Namely, the following theorems holds:

THEOREM 1. *If the function $F(x)$ is defined and continuous in the interval $[a, b]$, including the singular point c, the improper integral converges, and*

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{6.14}$$

i.e. the Newton–Leibniz formula holds true.

In applications, the most frequent cases encountered are those in which, when x tends to c from the right or from the left, the function $f(x)$ tends to infinity of a definite sign. On this condition, *if an improper integral converges, it converges absolutely*, i.e. the integral $\int_a^b |f(x)|\,dx$ converges as well.

The geometric sense of improper integrals of the type indicated is shown in Fig. 27.

If the improper integral converges, the areas enclosed between the graph of the function and its asymptote have finite values.

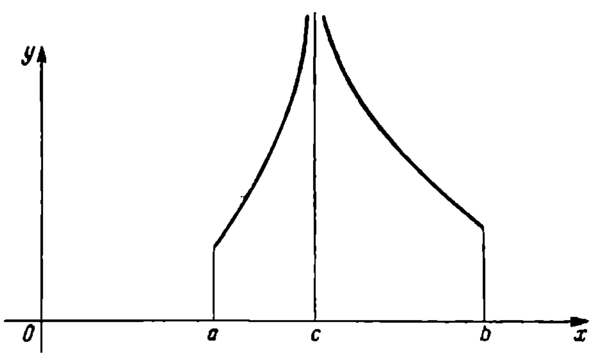

FIG. 27

The tests of convergence of integrals of unbounded functions are analogous to the corresponding tests for integrals with infinite limits. Since the definition of an improper integral in the case when c is an inner point of an interval demands the existence of integrals in both semi-open intervals $[a, c)$ and $(c, b]$, therefore in the formulation of tests of convergence it is sufficient to confine oneself to considering one of the cases. In the following, we shall assume that the singular point is at the right end.

CAUCHY TEST. For the integral $\int_a^c f(x)\,dx$ to converge, it is necessary and sufficient that to any number $\varepsilon > 0$ as small as desired, there corresponds a number δ, such that for all $\delta' < \delta$ and $\delta'' < \delta$ the following inequality holds

$$\left|\int_{c-\delta'}^{c-\delta''} f(x)\,dx\right| < \varepsilon. \tag{6.15}$$

COMPARISON TESTS. 1^0. If, excluding the singular point c,

$$|f(x)| \leqslant |\varphi(x)|,$$

then the absolute convergence of $\int_a^c f(x)\,dx$ follows from the absolute convergence of $\int_a^c \varphi(x)\,dx$; if, on the other hand, the integral of $f(x)$ is not absolutely convergent, the integral of $\varphi(x)$ also cannot be absolutely convergent.

2^0. If $\varphi(x)$ does not alter its sign when $x \to c$ and

$$\lim_{x\to c} \frac{f(x)}{\varphi(x)} = A \neq 0,$$

then both integrals simultaneously either converge or diverge.

3^0. If $f(x) = \psi(x)/(c - x)^\alpha$ and $\psi(x)$ is a function, continuous in the interval $[a, c]$, while $\psi(c) \neq 0$, the integral converges when $\alpha < 1$ and diverges when $\alpha \geqslant 1$.

If in the interval of integration, which may be infinite, there is a finite number of singular points, then the improper integral $\int_a^b f(x)\,dx$ is defined as a sum of improper integrals, taken over intervals, each of which contains only one singular point (if the end intervals are semi-infinite, it is assumed that there are no singular points in them). If the conditions of Cauchy's test are fulfilled in the neighbourhood of each singular point and at infinity, the integral is convergent and its value does not depend on the method of subdivision of the interval.

EXAMPLE 5. The integral

$$\int_0^\infty x^{a-1}e^{-x}\,dx \qquad (a > 0) \tag{6.16}$$

defines the so-called Euler's gamma-function $\Gamma(a)$. On representing it in the form $\int_0^1 + \int_1^\infty$, it is easy to verify the convergence of both integrals. The first integral converges because of the property 3°, and in the other one the integrand $x^{a-1}e^{-1} < 1/x^2$.

5. In view of the fact that the general properties of improper integrals refer to integrals of both types, in future the notation $\int_a^b f(x)\,dx$ is used, it being understood that a may equal $-\infty$ and b may equal $+\infty$.

The properties (a)–(g) of Chapter V, § 3, cover improper integrals without any alterations (it is assumed here, that all integrals present in the right-hand sides are convergent).

If $\int_a^b f(x)\,dx$ is convergent, then $\Phi(x) = \int_a^x f(x)\,dx$ is a continuous function in the interval $[a, b]$. If here $b = +\infty$ this means that there exists the limit $\lim_{x\to+\infty} \Phi(x) = \Phi(+\infty)$.

If the function $f(x)$ is continuous at the point x_0, then

$$\Phi'(x_0) = f(x_0).$$

The first theorem about mean value does not apply to improper integrals, and the second one holds as before in the following form:

If $f(x)$ is monotonic and bounded in the interval $[a, b]$ and $g(x)$ is integrable, then $f(x)\,g(x)$ is also integrable and

$$\int_a^b f(x)\,g(x)\,dx = f(a)\int_a^\xi g(x)\,dx + f(b)\int_\xi^b g(x)\,dx. \tag{6.17}$$

Integration by Parts. Change of Variable

6. Let $f(x)$ and $g(x)$ be two functions, continuous in the interval $[a, b]$, whose derivatives, $f'(x)$ and $g'(x)$, exist everywhere, except at a finite number of points, the derivatives having, perhaps, isolated discontinuities. Then the equation

$$\int_a^b f(x)\,\varphi'(x)\,dx = [f(x)\,\varphi(x)]_a^b - \int_a^b \varphi(x)\,f'(x)\,dx \tag{6.18}$$

(where a and b may become $-\infty$ and $+\infty$) holds, if any two of the three terms of the formula have a definite value.

EXAMPLE 6. The integral defining Euler's gamma-function $\Gamma(a) \int_0^\infty x^{a-1}e^{-x}dx$ is easily evaluated, when a is an integer ($a = n$),

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = -[x^{n-1}e^{-x}]_0^\infty + (n-1)\int_0^\infty x^{n-2}e^{-x}dx,$$

i.e.

$$\Gamma(n) = (n-1)\,\Gamma(n-1). \tag{6.19}$$

Continuing integration by parts we get

$$\Gamma(n) = (n-1)! \tag{6.20}$$

The formula

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,dt, \tag{6.21}$$

where $a = \varphi(\alpha)$ and $b = \varphi(\beta)$, holds when the derivative $\varphi'(t)$ is continuous and is non-zero inside the interval (α, β). At the ends of this interval both the function $\varphi(t)$ itself and its derivative $\varphi'(t)$ may have discontinuities.

The rule remains valid when α and β become infinite; here both $a = \lim\limits_{t\to\alpha}\varphi(t)$ and $b = \lim\limits_{t\to\beta}\varphi(t)$ may also become infinite. These limits, finite and infinite, always exist, since the function $\varphi(t)$ is monotonic.

EXAMPLE 7. By means of this rule, it is possible to establish by substitution the convergence of integrals of diffraction

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt;$$

the integral in the right-hand side converges on the basis of the analogue of Dirichlet's test.

7. In certain cases improper integrals can be defined (and sometimes calculated approximately) by means of infinite sums of certain forms.

Let $f(x)$ be continuous and monotonic in the interval (0, 1) whose ends may be singular points.

If the improper integral $\int_0^1 f(x)\,dx$ converges, then

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right). \tag{6.22}$$

If the function $f(x)$ is monotonic for $x \geqslant 0$ and the integral $\int_0^{\infty} f(x)\,dx$ converges, then

$$\int_0^{\infty} f(x)\,dx = \lim_{h\to+0} h \sum_{n=1}^{\infty} f(nh). \tag{6.23}$$

The Principal Value of an Integral

8. Suppose that the function $f(x)$ has one singular point c in the interval $[a, b]$.

If neither of the limits

$$\lim_{\delta\to 0}\int_a^{c-\delta} f(x)\,dx \quad \text{and} \quad \lim_{\delta'\to 0}\int_{c+\delta'}^{b} f(x)\,dx$$

exists, then $\int_a^b f(x)\,dx$ is called *singular.*

In many cases it is possible to give to this integral quite a definite meaning.

DEFINITION. The name *Cauchy principal value of the singular integral* is given to the limit (if it exists)

$$\lim_{\delta\to 0}\left\{\int_a^{c-\delta} f(x)\,dx + \int_{c+\delta}^{b} f(x)\,dx\right\}. \tag{6.24}$$

The principal value of the singular integral is usually denoted by the same symbol as the integral itself:

$$\int_a^b f(x)\,dx,$$

so that, if the integral is proper or improper, but convergent, its principal value coincides with the value of the integral itself.

Note. In earlier works the notation used for the principal value of a singular integral was the symbol $p \int$ or the symbol v.p. $\int$ (valeur principale).

The principal value of the singular integral cannot exist if the function $f(x)$ tends to infinity of *one sign* for $x \to c \pm 0$. Also, if *only one* of the two limits does not exist, then neither does the limit defining the principal value of the integral.

It is usual to consider only those functions which tend to infinity of a definite sign when $x \to c$ from the left and from the right (Fig. 28).

In this case, the sense of the principal value of the integral consists in just this fact, that both the terms of the sum tend to infinity of different signs, but their sum tends to quite a definite limit.

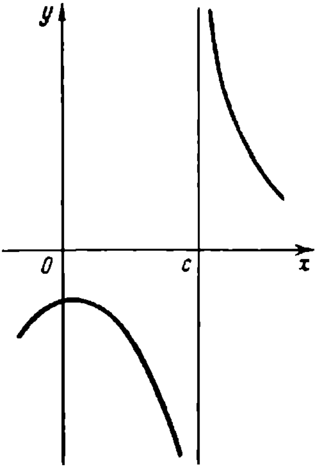

FIG. 28

As an example, let us consider the singular integral

$$\int_a^b \frac{dx}{x-c}, \quad a < c < b.$$

This integral diverges in the usual sense, since

$$\int_a^{c-\delta'} \frac{dx}{x-c} + \int_{c+\delta''}^b \frac{dx}{x-c} = \ln \frac{b-c}{c-a} + \ln \frac{\delta'}{\delta''}$$

and the limit of the second term depends on the way in which δ' and δ'' tend to zero. But the principal value of the integral does exist, since

$$\lim_{\delta \to 0} \left[\int_a^{c-\delta} \frac{dx}{x-c} + \int_{c+\delta}^b \frac{dx}{x-c} \right] = \ln \frac{b-c}{c-a}.$$

In the same way, it can be proved that the principal value of the integral $\int_a^b \frac{dx}{(x-c)^n}$ exists if *n is an odd* number. If, on the other hand, n is an *even* number, the principal value of the integral does not exist, since for $x \to c \pm 0$, the function $1/(x-c)^n$ tends to $+\infty$.

In applications, integrals of the type

$$\int_a^b \frac{\varphi(x)}{x-c}\,dx$$

are often encountered. If the function $\varphi(x)$ satisfies Hölder's condition† then the principal value of this integral exists and

$$\int_a^b \frac{\varphi(x)}{x-c}\,dx = \int_a^b \frac{\varphi(x)-\varphi(c)}{x-c}\,dx + \varphi(c)\ln\frac{b-c}{c-a},$$

where the improper integral on the right converges.

Similarly, the principal value of the integral $\int_{-\infty}^{+\infty} f(x)\,dx$ is defined as the limit (if it exists)

$$\lim_{N\to\infty}\int_{-N}^{N} f(x)\,dx.$$

Here it is assumed that the function $f(x)$ is integrable in any finite interval.

It is often necessary to consider integrals in which the principal value is understood in both senses at the same time.

For example

$$\int_{-\infty}^{+\infty}\frac{dx}{x-c} = \lim_{\substack{N\to\infty\\ \delta\to 0}}\left[\int_{-N}^{c-\delta} + \int_{c+\delta}^{N}\right] = 0.$$

Here the following sufficient condition of existence of the principal value of the integral $\int_{-\infty}^{+\infty}\frac{f(x)}{x-c}\,dx$ is present.

If $f(x)$ satisfies Hölder's condition in the neighbourhood of the point c, and if it has the following form for large values of $|x|$:‡

$$f(x) = A + O\left(\frac{1}{|x|^{\mu}}\right) \quad (\mu > 0), \tag{6.25}$$

the principal value of the integral does exist.

9. *Improper double integrals* are obtained in a similar way to ordinary ones if infinite domains of integration or unbounded functions are considered; integrals taken in the sense of § 4 are called *proper* from now on.

† The function $\varphi(x)$ is said to satisfy *Hölder's condition* in the interval $[a, b]$ if for any two points x_1 and x_2 of this interval

$$|\varphi(x_2) - \varphi(x_1)| < A\,|x_2 - x_1|^{\alpha},$$

where $A > 0$ and $0 < \alpha \leqslant 1$. Sometimes this condition is known as Lipshitz's condition of order α and it is written down as $\varphi(x) \in \mathrm{Lip}\,\alpha$.

‡ It follows hence, among other things, that, for $x \to +\infty$ and for $x \to -\infty$, the function $f(x)$ tends to the same limit A.

Let the domain of integration D extend to infinity in all directions, or, at least, in some directions (for example, a strip or a sector bounded by two straight lines), and the function $f(x, y)$ be *non-negative* in this domain and *properly* integrable in any finite part of it.

The name *improper integral* $\displaystyle\int\int_D f(x,y)\,dx,dy$ is given to the limit of the proper integral taken in any finite part D' of the region D on condition that D' tends to D:

$$\int\int_D f(x, y)\,dx\,dy = \lim_{D'\to D}\int\int_{D'} f(x, y)\,dx\,dy. \tag{6.26}$$

Here D' can tend to D in any way; either in all directions at once or in the various directions in succession. It follows from the non-negativeness of the function $f(x, y)$ that if a limit (finite or infinite) exists for any one sequence of regions $\{D'\}$, it also exists for any other sequence and these limits are all equal. If this limit is finite, the integral is called *convergent*; if the opposite is the case it is called *divergent*.

The case of the non-positive function is reduced to the preceding one by change of sign. Suppose the function $f(x, y)$ *does not preserve* a constant sign. Then the integral

$$\int\int_D f(x, y)\,dx\,dy$$

is considered as existing only when it converges absolutely, i.e. when the integral

$$\int\int_D |f(x, y)|\,dx\,dy$$

exists. If this condition is satisfied, then the limiting process can be completed in an arbitrary fashion.

The idea of conditional convergence does not extend to improper double integrals. Put $f(x, y) = f^+ = f^-$ where $f^+ = f$ if $f > 0$ and $f^+ = 0$ if $f \leqslant 0$; similarly, $f^- = -f$ if $f < 0$ and $f^- = 0$ if $f \geqslant 0$. Then,

$$\int\int_D f(x, y)\,dx\,dy = \int\int_D f^+(x, y)\,dx\,dy - \int\int_D f^-(x, y)\,dx\,dy, \tag{6.27}$$

and

$$\int\int_D |f(x, y)|\, dx\, dy = \int\int_D f^+ dx\, dy + \int\int_D f^- dx\, dy. \quad (6.28)$$

If integrals of functions f^+ and f^- exist the integral converges, and absolutely at that. If one of the integrals shown diverges (here it must do so to infinity) the given integral also diverges to infinity (of a definite sign). If the integrals of both the functions f^+ and f^- diverge, their difference is *indeterminate*. If the function $f(x, y)$ is bounded, then it is always possible to select a sequence of regions $D' \to D$, such that the limit of $\int\int_{D'} f(x, y)\, dx\, dy$ can be made equal to any given number. (The *analogue of Riemann's theorem* for conditionally converging series.)

Suppose now that the domain of integration D is finite and bounded by the squarable contour C, and the function $f(x, y)$ is *non-negative* and has points or lines of discontinuity which satisfy the conditions given in § 4 of Chapter V, but in whose neighbourhood the function can no longer be unbounded. Such points of discontinuity are called as before, *singular* points. It is put by definition

$$\int\int_D f(x, y)\, dx\, dy = \lim_{D' \to D} \int\int_{D'} f(x, y)\, dx\, dy, \quad (6.29)$$

where the region D' is obtained from the region D by the exclusion of all singular points and lines; here, isolated singular points and lines of discontinuity which do not subdivide the region D are encircled by closed contours, and lines of discontinuity which do subdivide the region are enclosed by two infinitely close contours. Just as in the case of the infinite region, it follows from the non-negativeness of the function $f(x, y)$ that the limit in the right-hand side does not depend on the way in which D' tends to D and for this limit to exist it is sufficient that the integral over any region of D' be *bounded.*

If the function $f(x, y)$ is of alternating sign, then its improper double integral is regarded as existing only when $\int\int_D |f(x, y)|\, dx\, dy$ is convergent. The idea of conditional convergence does not extend to improper integrals of finite regions either.

EXAMPLE 8. Consider the integral

$$\iint_D \frac{y^2 - x^2}{(y^2 + x^2)^2}\, dx\, dy,$$

where D is a rectangle, bounded by straight lines $x = 0$, $x = a$, $y = 0$, $y = b$ $(a > 0,\ b > 0)$. Let us exclude the origin by cutting off a small rectangle by means of straight lines $x = \alpha$ and $x = \beta$. Then, denoting the remaining region by D', we find that

$$\iint_{D'} \frac{y^2 - x^2}{(y^2 + x^2)^2}\, dx\, dy = \operatorname{arctg}\frac{b}{a} - \operatorname{arctg}\frac{\beta}{\alpha}.$$

The limit of this double integral depends on the limit of the ratio β/α when α and β tend to zero, i.e. the integral diverges.

The reduction of improper double integrals to repeated ones in both cases considered above is based on the following rule:

If the function $f(x, y)$ does not change sign in the region D, then the improper double integral is reducible to the repeated integral in accordance with the usual rule, provided the integration in one variable leads to a function of the second variable, which is integrable properly or improperly.

If $f(x, y)$ changes sign in the region D, then, in order to check the validity of the reduction to a repeated integral, it is sufficient to subdivide the region D into parts, in each of which the function is of constant sign, and to verify that the conditions of the given rule are satisfied in each of them.

The formula of transformation of double integrals

$$\iint_D f(x, y)\, dx\, dy = \iint_\Delta |I|\, \Phi(u, v)\, du\, dv \tag{6.30}$$

has the same sense, as formula (5.110), i.e. if one of the integrals exists then the other one exists also, and they are equal. Here, the regions D and Δ may be either finite or infinite.

By means of transformation to polar coordinates, it can be established, that, if the region D is infinite, does not contain the origin and the integrand has the form

$$f(x, y) = \frac{\psi(x, y)}{(x^2 + y^2)^\alpha}, \quad 0 < m < |\psi(x, y)| < M, \tag{6.31}$$

then the integral converges for $\alpha > 1$ and diverges for $\alpha \leqslant 1$. Conversely, if the region of integration is finite and the function has

only one singular point, for example, the origin, in whose neighbourhood the function has the form (6.31) the integral converges for $\alpha < 1$ and diverges for $\alpha \geqslant 1$.

If the function $\psi(x, y)$ is required to be bounded only, i.e. $|\psi(x, y)| < M$, the integral converges, when $\alpha > 1$ in the first case and $\alpha < 1$ in the second.

Using these tests it can be proved, for example, that the following integrals converge:

$\iint_D e^{-x^2-y^2}\, dx\, dy$ taken over any infinite region;

$\iint_D (x + y)\, e^{-(x+y)}\, dx\, dy$ taken in the first quadrant $x > 0$, $y > 0$;

the integral $\iint_D \ln\sqrt{x^2 + y^2}\, dx\, dy$ taken in the circle $x^2 + y^2 \leqslant 1$ is convergent and the integral $\displaystyle\iint_D \frac{e^{-x^2-y^2}}{x^2 + y^2}\, dx\, dy$, in the same region, diverges.

EXAMPLE 9. Let us calculate the integral $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$ in polar coordinates.

Selecting, as a sequence of regions D', circles of radius R, we obtain

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} e^{-x^2-y^2} dy = \lim_{R\to\infty} \left[\int_0^{2\pi} d\varphi \int_0^R e^{-r^2} r\, dr\right] = \pi.$$

Taking regions D' in the form of squares $|x| \leqslant a$, $|y| \leqslant a$, we find from the independence of the limiting process, that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \pi,$$

i.e.

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{6.32}$$

The latter integral plays an important part in the theory of probability and is called ***Poisson's integral***.

If the function $f(x, y)$ becomes infinite on approaching the boundary of the region, then it is frequently possible to make use of the following test:

If the function $f(x, y)$ is continuous in the region D: $a \leqslant x \leqslant b$, $0 \leqslant y \leqslant g(x)$, where $g(x)$ is a continuous function in the interval $[a, b]$, then the integral

$$\iint_D \frac{\psi(x, y)}{[g(x) - y]^\alpha}\, dx\, dy, \quad 0 < m < |\psi(x, y)| < M,$$

converges for $\alpha < 1$ and diverges for $\alpha \geqslant 1$.

Improper multiple integrals, where the number of variables is greater than two, are defined in a way similar to the double ones. We note the sufficient conditions for the convergence of these integrals.

The integral

$$\underbrace{\int\int\cdots\int}_{n} \frac{\psi(x_1, x_2, \ldots, x_n)}{(x_1^2 + x_2^2 + \cdots x_n^2)^\alpha}\, dx_1\, dx_2 \ldots dx_n, \tag{6.33}$$

where $|\psi(x_1, x_2, \ldots, x_n)| < M$, *taken over an infinite region, not containing the origin of the coordinates, converges if* $\alpha > n/2$; *the same integral, taken over any finite region containing the origin, converges if* $\alpha < n/2$.

§ 2. The Limiting Process under the Sign of the Integral. Integrals Dependent on a Parameter

1. Let $\{f_n(x)\}$ be a sequence of functions all of which are defined in the same interval $[a, b]$. Is it possible to conclude from the convergence of this sequence, that the sequences $\left\{\int_a^b f_n(x)\, dx\right\}$ or $\left\{\int_a^x f_n(t)\, dt\right\}$ $(a \leqslant x \leqslant b)$ converge or otherwise? If the sequences $\{f_n(x)\}$ and $\left\{\int_a^b f_n(x)\, dx\right\}$ converge, is there a connection between their limits? These problems are usually known as problems about the possibility of the limiting process under the integral sign.

Simple examples show that the convergence of the sequence $\{f_n(x)\}$ at every point of the interval does not secure the convergence of the sequence of integrals $\left\{\int_a^b f_n(x)dx.\right\}$

Example 10. Let us integrale the terms of the sequence $\{f_n(x)\}$ in the interval $[0, 1]$, as shown in Fig. 29, i.e.

$$f_n(x) = \begin{cases} n^3x & \text{when} \quad 0 \leqslant x \leqslant \dfrac{1}{n}, \\ 2n^2 - n^3x & \text{when} \quad \dfrac{1}{n} < x < \dfrac{2}{n}, \\ 0 & \text{when} \quad \dfrac{2}{n} \leqslant x \leqslant 1. \end{cases}$$

The sequence $\{f_n(x)\}$ converges at every point of the interval [0, 1], because for any point x, except $x = 0$, there can be found N, such that for all $n > N$ we have $x > 2/n$, whence it follows, that $f_n(x) = 0$ for $n > N$. Thus $\lim\limits_{n\to\infty} f_n(x) = 0$ which holds also for $x = 0$, since $f_n(0) = 0$ for all n.

On the other hand, the integral

$$\int_0^1 f_n(x)\,dx = \int_0^{\frac{1}{n}} n^3x\,dx + \int_{\frac{1}{n}}^{\frac{2}{n}} (2n^2 - n^3x)\,dx = n,$$

whence it follows that the sequence $\left\{\int_0^1 f_n(x)\,dx\right\}$ diverges.

It is just as simple to single out the example of the sequence $\{f_n(x)\}$ converging at every point, for which the sequence of integrals $\left\{\int_a^b f_n(x)\,dx\right\}$ converges but

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx \neq \int_a^b \left\{\lim_{n\to\infty} f_n(x)\right\} dx.$$

EXAMPLE 11. Suppose we are given a sequence of functions, defined as in Fig. 29, but with value n and not n^2 at the point $1/n$, i.e.

$$f_n(x) = \begin{cases} n^2x & \text{when} \quad 0 \leqslant x \leqslant \dfrac{1}{n}, \\ 2n - n^2x & \text{when} \quad \dfrac{1}{n} < x < \dfrac{2}{n}, \\ 0 & \text{when} \quad \dfrac{2}{n} \leqslant x \leqslant 1. \end{cases}$$

As above, it is easy to prove that $\lim\limits_{n\to\infty} f_n(x) = 0$ at every point of the interval [0, 1] and, therefore,

$$\int_0^1 \{\lim_{n\to\infty} f_n(x)\}\,dx = 0.$$

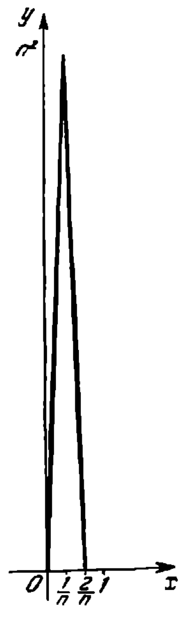

FIG. 29

However,

$$\int_0^1 f_n(x)\,dx = \int_0^{\frac{1}{n}} n^2x\,dx + \int_{\frac{1}{n}}^{\frac{2}{n}} (2n - n^2x)\,dx = 1,$$

as a consequence of which $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$.

Thus, the convergence of the sequence $\{f_n(x)\}$ at every point of an interval does not guarantee the admissibility of the limiting

process under the integral sign; more rigid requirements are necessary here.

For Riemann's integral the possibility of the limiting process under the integral sign is assured by uniform convergence.

THEOREM 2. *If a sequence of functions* $\{f_n(x)\}$, *continuous in the interval* $[a, b]$, *converges uniformly in this interval to the function* $f(x)$, *then the sequence of functions* $\left\{\int_a^x f_n(t)\,dt\right\}$ *converges uniformly to the function* $\int_a^x f(t)\,dt$ *in the interval* $[a, b]$.

If we regard the functions $f_n(x)$ as partial sums of the functional series $\sum_{n=1}^{\infty} u_n(x)$, we can deduce the following proposition.

THEOREM 3. *If the functions of the series* $\{u_n(x)\}$ *are continuous when* $a \leqslant x \leqslant b$ *and the series* $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly in* $[a, b]$ *to the function* $f(x)$, *then the series*

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt \tag{6.34}$$

also converges uniformly for $a \leqslant x \leqslant b$, *and*

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)\,dt = \int_a^x f(t)\,dt \quad (a \leqslant x \leqslant b). \tag{6.35}$$

In particular,

$$\sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx = \int_a^b f(x)\,dx. \tag{6.36}$$

Since a series of positive continuous functions having a continuous sum always converges uniformly, a series of this kind can always be integrated term by term.

2. The integrals of the functions of the sequence $\int_a^b f_n(x)\,dx$ are a particular case of *integrals* $\int_a^b f(x, \alpha)\,dx$, *dependent on a parameter*.

Suppose that $f(x, \alpha)$ is a function of the variable x and the parameter α, and is defined in the closed rectangle $a \leqslant x \leqslant b$, $\alpha_0 \leqslant \alpha \leqslant \alpha_1$. On integrating this function with respect to the variable x

over the interval $[a, b]$ we obtain an integral, which is known as *an integral dependent on a parameter.*

The integral

$$\int_a^b f(x, \alpha)\, dx = F(\alpha), \tag{6.37}$$

dependent on a parameter is a function of the parameter α.

An important method of representing a function analytically is one whereby the function is given by an integral dependent on a parameter.

THEOREM 4. *If the function* $f(x, \alpha)$ *is continuous in the closed rectangle* $a \leqslant x \leqslant b, \alpha_0 \leqslant \alpha \leqslant \alpha_1$, *then the integral* $F(\alpha) = \int_a^b f(x, \alpha)\, dx$, *dependent on a parameter, is a continuous function of the parameter* α *in the interval* $[\alpha_0, \alpha_1]$.

The possibility of differentiation of an integral with respect to a parameter is established by the following theorem.

THEOREM 5 *(Leibniz's rule). If the function* $f(x, \alpha)$ *has a partial derivative in* α, *continuous in the closed rectangle* $a \leqslant x \leqslant b$, $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, *then the integral* $F(\alpha) = \int_a^b f(x, \alpha)\, dx$, *dependent on a parameter, is a differentiable function of parameter* α, *and its derivative can be obtained by means of differentiating with respect to the parameter under the sign of the integral.*

In other words, the following formula holds:

$$F'(\alpha) = \int_a^b f'_\alpha(x, \alpha)\, dx. \tag{6.38}$$

The application of Leibniz's rule makes it easier to calculate certain definite integrals.

EXAMPLE 12. It is known that

$$\int_0^1 \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{1}{a}.$$

Regarding a as a parameter, and differentiating this equation with respect to a (here the integral on the left is differentiated with respect to parameter a under the integral sign), we find

$$\int_0^1 \frac{-2a\, dx}{(x^2 + a^2)^2} = -\frac{1}{a^2} \arctan \frac{1}{a} - \frac{1}{a} \frac{1}{1 + \frac{1}{a^2}} \frac{1}{a^2},$$

whence
$$\int_0^1 \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\arctan\frac{1}{a} + \frac{1}{2a^2(a^2+1)}.$$

Similarly, we find after the following differentiation:
$$\int_0^1 \frac{dx}{(x^2+a^2)^3} = \frac{3}{8a^5}\arctan\frac{1}{a} + \frac{3}{8a^4(a^2+1)} + \frac{1}{4a^2(a^2+1)^2},$$

etc.; thus, by means of differentiation in a, it is possible to obtain the value of the integral $\int_0^1 \frac{dx}{(x^2+a^2)^n}$ for any whole n.

Leibniz's rule can be extended to include a more complex case, when the limits of integration also depend on the parameter α.

THEOREM 6. *If*
$$F(\alpha) = \int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx,$$

where the function $f(x,\alpha)$ has a partial derivative in α, continuous for $\varphi(\alpha) \leqslant x \leqslant \psi(\alpha)$, $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, and the functions $\varphi(\alpha)$ and $\psi(\alpha)$ are continuous and are continuously differentiable for $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, then the function $F(\alpha)$ is differentiable and
$$F'(\alpha) = \int_{\varphi(\alpha)}^{\psi(\alpha)} f'_\alpha(x,\alpha)\,dx + f[\psi(\alpha),\alpha]\,\psi'(\alpha) - f[\varphi(\alpha),\alpha]\,\varphi'(\alpha). \tag{6.39}$$

The possibility of integration with respect to the parameter under the sign of the integral is ensured by the existence of the double integral of the function $f(x,\alpha)$ over the rectangle being considered (see § 4). In this case
$$\int_{\alpha_0}^{\alpha_1}\left(\int_a^b f(x,\alpha)\,dx\right)d\alpha = \int_a^b\left(\int_{\alpha_0}^{\alpha_1} f(x,\alpha)\,d\alpha\right)dx. \tag{6.40}$$

3. The rules and propositions formulated in § 2 hold, without any additional provisos, only for *proper* (see § 1) integrals. For integrals with infinite limits, or for functions unbounded in the domain of integration, additional assumptions are necessary.

In the case of improper integrals, dependent on a parameter, with infinite limits, it is possible, as was shown in § 1, to confine oneself to the consideration of an integral with an infinite *upper* limit only:
$$F(\alpha) = \int_a^\infty f(x,\alpha)\,dx. \tag{6.41}$$

Suppose the function $f(x, \alpha)$ is defined for $x \geqslant a$, $\alpha_0 \leqslant \alpha \leqslant \alpha_1$ and the improper integral $F(\alpha)$ exists for all α in the interval $[\alpha_0, \alpha_1]$. The improper integral $\int_a^\infty f(x, \alpha)\, dx$ is said to be *uniformly convergent with respect to the parameter α in the interval* $[\alpha_0, \alpha_1]$ if for any $\varepsilon > 0$ there exists such a number $A \geqslant a$ such that

$$\left| \int_c^\infty f(x, \alpha)\, dx \right| < \varepsilon \tag{6.42}$$

for all $c > A$ for any α in the interval $[\alpha_0, \alpha_1]$.

The definition of a uniformly convergent integral is analogous to the definition of a uniformly convergent series. Moreover, *a uniformly convergent improper integral can be represented in the form of the sum of a series of uniformly convergent proper integrals.* This analogy helps to establish the following properties of uniformly convergent improper integrals.

(a) *If the improper integral*

$$F(\alpha) = \int_a^\infty f(x, \alpha)\, dx, \tag{6.43}$$

where $f(x, \alpha)$ is defined and continuous for $x \geqslant a$, $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, exists and converges uniformly in the interval $[\alpha_0, \alpha_1]$, then it is a continuous function of the parameter α.

(b) *Given the same assumptions, the following equation holds*:

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_a^\infty f(x, \alpha)\, dx = \int_a^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\, d\alpha, \tag{6.44}$$

and the existence of the right-hand integral is guaranteed.

(c) *If the function $f(x, \alpha)$ satisfies the preceding conditions and possesses a partial derivative in α, continuous for $x \geqslant a$, $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, and the improper integral $\int_a^\infty f'_\alpha(x, \alpha)\, dx$ exists and converges uniformly, then the improper integral* (6.43) *dependent on the parameter has in the interval $[\alpha_0, \alpha_1]$ a continuous derivative in α, which can be obtained by differentiation with respect to the parameter under the integral sign*:

$$F'(\alpha) = \int_a^\infty f'_\alpha(x, \alpha)\, dx. \tag{6.45}$$

In order to establish the uniform convergence of an improper integral it is sufficient to make use of the following theorem.

THEOREM 7. *If there exists a function* $\varphi(x)$, *for which the integral* $\int_a^\infty |\varphi(x)|\,dx$ *converges, and if for all* $x \geqslant a$ *and for all* α *of the interval* $[\alpha_0, \alpha_1]$ *the equation* $|f(x, \alpha)| \leqslant |\varphi(x)|$ *holds, then the improper integral* $\int_a^\infty f(x, \alpha)\,dx$ *converges uniformly with respect to* α *in the interval* $[\alpha_0, \alpha]$.

EXAMPLE 13. The integral $\int_0^\infty e^{-x} \sin \alpha x\,dx$ converges uniformly with respect to α in any interval because

$$|e^{-x} \sin \alpha x| \leqslant e^{-x}$$

and the integral $\int_0^\infty e^{-x}\,dx$ exists, as was shown in § 1. By means of direct integration we get

$$\int_0^\infty e^{-x} \sin \alpha x\,dx = \frac{\alpha}{1+\alpha^2}. \tag{6.46}$$

Since the integral converges uniformly, it can be integrated with respect to the parameter α within the limits from 0 to y. Then

$$\int_0^y d\alpha \int_0^\infty e^{-x} \sin \alpha x\,dx = \int_0^\infty dx \int_0^y e^{-x} \sin \alpha x\,d\alpha = \int_0^\infty e^{-x} \frac{1-\cos xy}{x}\,dx,$$

whence

$$\int_0^\infty e^{-x} \frac{1-\cos xy}{x}\,dx = \frac{1}{2} \ln(1+y^2). \tag{6.47}$$

The differentiation of the original integral with respect to α gives the integral $\int_0^\infty x\,e^{-x} \cos \alpha x\,dx$, which also converges uniformly, since $|x\,e^{-x} \cos \alpha x| \leqslant x\,e^{-x}$ and the integral $\int_0^\infty x\,e^{-x}\,dx$ exists. Therefore the differentiation with respect to the parameter under the integral sign is possible and

$$\int_0^\infty x e^{-x} \cos \alpha\,dx = \frac{1-\alpha^2}{(1+\alpha^2)^2}. \tag{6.48}$$

EXAMPLE 14. Suppose it is required to calculate the integral $\int_0^\infty \frac{dx}{x^2+\alpha^2}$. Since $\frac{1}{x^2+\alpha^2} \leqslant \frac{1}{x^2+1}$ and the integral $\int_0^\infty \frac{dx}{x^2+1}$ converges, therefore our integral converges uniformly. Direct calculation gives

$$\int_0^\infty \frac{dx}{x^2+\alpha^2} = \frac{\pi}{2\alpha}. \tag{6.49}$$

Integrating in α from 1 to y we find

$$\int_0^\infty \frac{\arctan\dfrac{y}{x} - \arctan\dfrac{1}{x}}{x}\,dx = \frac{\pi}{2}\ln y. \tag{6.50}$$

Indeed

$$\int_1^y d\alpha \int_0^\infty \frac{dx}{x^2+\alpha^2} = \int_0^\infty dx \int_1^y \frac{d\alpha}{x^2+\alpha^2} = \int_0^\infty \left[\frac{1}{x}\arctan\frac{\alpha}{x}\right]_{\alpha=1}^{\alpha=y} dx$$

$$= \int_0^\infty \frac{\arctan\dfrac{y}{x} - \arctan\dfrac{1}{x}}{x}\,dx.$$

4. All definitions and theorems of § 3 are easily applied to the case of improper integrals of unbounded functions. In future, we consider only the case in which the function $f(x, \alpha)$ becomes infinite in a finite number of points x of the interval $[a, b]$ independent of α, i.e. either in isolated points of the rectangle $a \leqslant x \leqslant b$, $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, or along segments of straight lines parallel to the α-axis. In the remaining points of the rectangle, in whose neighbourhood the function $f(x, \alpha)$ is bounded, we assume it to be continuous.

Suppose, to begin with, that the function $f(x, \alpha)$ can become infinite only when $x = b$. Then the improper integral $\int_a^b f(x, \alpha)\,dx$ is said to be *uniformly convergent with respect to α in the interval* $[\alpha_0, \alpha_1]$, if, for any $\varepsilon > 0$, it is possible to find such a $\delta > 0$ independent of α, that for all numbers η, satisfying the condition $0 < \eta < \delta$, and for all values of α considered, the following inequality holds,

$$\left|\int_{b-\eta}^b f(x, \alpha)\,dx\right| < \varepsilon.$$

If $f(x, \alpha)$ can become infinite only for $x = a$, then, in place of the latter inequality, the following inequality should hold

$$\left|\int_a^{a+\eta} f(x, \alpha)\,dx\right| < \varepsilon.$$

If several discontinuities are present, the interval $[a, b]$ should be subdivided into segments in such a way that in each of them $f(x, \alpha)$ will have only one point at which it can become infinite. If

each of the constituent integrals converges uniformly, the same can be said of the given integral.

The properties of uniformly convergent improper integrals dependent on a parameter given in sec. 3 apply also to the case of integrals of unbounded functions.

(a) *An improper integral dependent on a parameter α is a continuous function of this parameter in every interval in which it converges uniformly. If the integrand is positive, then in order that the integral with respect to the parameter be continuous it is necessary and sufficient that it converges uniformly.*

(b) *Integration with respect to parameter α under the sign of the improper integral is possible in any finite interval in which there is uniform convergence. The integral thus obtained is also uniformly convergent with respect to the parameter α.*

(c) *If differentiation with respect to the parameter under the integral sign leads to a uniformly convergent improper integral, then Leibniz's rule holds (Theorem 5 of § 2).*

§ 3. The Stieltjes Integral for Functions of One Variable

1. Let the functions $f(x)$ and $\varphi(x)$ be defined in the interval $[a, b]$. We subdivide this interval into n elementary parts and, taking in each an arbitrary point ξ_k, we construct the *Stieltjes integral sum*

$$\sigma_n = \sum_{k=1}^{n} f(\xi_k)\Delta_k\varphi(x), \tag{6.51}$$

where $\Delta_k \varphi(x) = \varphi(x_k) - \varphi(x_{k-1})$ is the increment of the function $\varphi(x)$ in the kth elementary part.

The limit of Stieltjes' integral sums, when the lengths of all elementary divisions tend to zero, is called *the Stieltjes integral of the function $f(x)$ with respect to the function $\varphi(x)$*:

$$\int_a^b f(x)\, d\varphi(x) = \lim_{\substack{n\to\infty \\ \max \Delta_k x \to 0}} \sum_{k=1}^{n} f(\xi_k)\Delta_k\varphi(x). \tag{6.52}$$

The Stieltjes integral is a generalization of the Riemann integral; the increment of some *integrating function* $\varphi(x)$ in the interval is taken as the measure of that interval (see Chapter V, § 3).

If the above limit exists, it is said that the function $f(x)$ is *integrable with respect to the function* $\varphi(x)$. The existence of Stieltjes' integral is established by means of the following theorems:

THEOREM 8. *If the function $f(x)$ is continuous and $\varphi(x)$ is of finite variation*† *then the integral* $\int_a^b f(x)\,d\varphi(x)$ *exists.*

THEOREM 9. *If the function $f(x)$ is integrable in the interval $[a, b]$ in the Riemann sense, and the function $\varphi(x)$ satisfies the Lipshitz condition*

$$|\varphi(x_2) - \varphi(x_1)| < K\,|x_2 - x_1|$$

(x_1, x_2 are arbitrary points of the interval $[a, b]$ and K is a fixed constant), then the function $f(x)$ is integrable with respect to the function $\varphi(x)$.

THEOREM 10. *If $f(x)$ is Riemann-integrable and the function $\varphi(x)$ is differentiable and has a derivative, which is integrable in $[a, b]$, then $f(x)$ is integrable with respect to $\varphi(x)$ and*

$$\int_a^b f(x)\,d\varphi(x) = (R)\int_a^b f(x)\,\varphi'(x)\,dx. \tag{6.53}$$

The latter theorem can also be used to calculate a Stieltjes integral by reducing it to a Riemann integral.

If $f(x)$ is integrable in the interval $[a, b]$ with respect to the function, then, conversely, $\varphi(x)$ is also integrable with respect to $f(x)$.

2. The Stieltjes integral possesses properties which are analogous to those of the definite Riemann integral. In formulating these properties, it is assumed that all the integrals considered exist.

(a)
$$\int_a^b d\varphi(x) = \varphi(b) - \varphi(a). \tag{6.54}$$

† A function $f(x)$ is said to be of ***bounded variation*** in the interval $[a, b]$ if there exists a number $M > 0$ such that, for any set of points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of this interval, the following inequality holds:

$$\sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)| < M.$$

(b) *If c and k are constants, then*

$$\int_a^b cf(x)\,dk\varphi(x) = ck\int_a^b f(x)\,d\varphi(x). \tag{6.55}$$

(c) $$\int_a^b [f(x) \pm g(x)]\,d\varphi(x) = \int_a^b f(x)\,d\varphi(x) \pm \int_a^b g(x)\,d\varphi(x). \tag{6.56}$$

(d) $$\int_a^b f(x)\,d[\varphi(x) \pm \psi(x)] = \int_a^b f(x)\,d\varphi(x) \pm \int_a^b f(x)\,d\psi(x). \tag{6.57}$$

(e) *If the integral* $\int_a^b f(x)\,d\varphi(x)$ *exists and* $a < c < b$, *then*

$$\int_a^b f(x)\,d\varphi(x) = \int_a^c f(x)\,d\varphi(x) + \int_c^b f(x)\,d\varphi(x); \tag{6.58}$$

here the integrals on the right exist. It is necessary to keep in mind that the existence of the integral on the left does not follow from the existence of the integrals on the right, i.e. the function may be integrable in two parts of the interval, without it being integrable over the whole interval.

(f) THEOREM 11. *If the function* $f(x)$ *satisfies the inequalities* $m \leqslant f(x) \leqslant M$ *in the interval* $[a, b]$ *and is integrable with respect to the increasing function* $\varphi(x)$, *then*

$$\int_a^b f(x)\,d\varphi(x) = \mu[\varphi(b) - \varphi(a)], \tag{6.59}$$

where $m < \mu < M$.

If $f(x)$ is continuous, there exists a point ξ of the interval $[a, b]$, for which $f(\xi) = \mu$, i.e.

$$\int_a^b f(x)\,d\varphi(x) = f(\xi)\,[\varphi(b) - \varphi(a)]. \tag{6.60}$$

(g) THEOREM 12. *If the function* $f(x)$ *is continuous in the interval* $[a, b]$ *and* $\varphi(x)$ *is of bounded variation in it then*

$$\left|\int_a^b f(x)\,d\varphi(x)\right| \leqslant MV, \tag{6.61}$$

where $M = \max\limits_{a \leqslant x \leqslant b} |f(x)|$ and V is the complete variation of the function $\varphi(x)$.

3. The conditions for the limiting process under the Stieltjes integral differ little from the conditions for the limiting process of the Riemann integral, which were considered in § 2, sec. 1. They can be given in the form of two theorems.

THEOREM 13. *If the sequence* $\{f_n(x)\}$ *of continuous functions converges uniformly in* $[a, b]$ *to the function* $f(x)$, *and the function* $\varphi(x)$ *is of finite variation, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, d\varphi(x) = \int_a^b f(x)\, d\varphi(x), \tag{6.62}$$

and the existence of the limit of the sequence

$$\left\{\int_a^b f_n(x)\, d\varphi(x)\right\}$$

is guaranteed.

THEOREM 14. *If the function* $f(x)$ *is continuous in* $[a, b]$ *and the sequence of functions* $\{\varphi_n(x)\}$, *of bounded variation, converges to the function* $\varphi(x)$, *and the total variations of functions* $\varphi_n(x)$ *are bounded in the set, then*

$$\lim_{n\to\infty} \int_a^b f(x)\, d\varphi_n(x) = \int_a^b f(x)\, d\varphi(x), \tag{6.63}$$

and the existence of a limit on the left is guaranteed.

4. Stieltjes' integral is useful in finding static moments, moments of inertia or bending moments of masses, distributed along a segment $[a, b]$, if in addition to a continuous distribution, there are masses concentrated in separate points.

Let $\varphi(x)$ denote the sum of all masses in the interval $[a, x]$. Then, for the static moment M with respect to the axis Oy we have

$$M = \int_a^b x\, d\varphi(x). \tag{6.64}$$

The moment of inertia I of the interval $[a, b]$ of the axis Ox with respect to the axis Oy equals

$$I = \int_a^b x^2\, d\varphi(x). \tag{6.65}$$

Similarly, for a beam supported at the ends $(0, l)$ (Fig. 30), the bending moment M acting at the section ξ equals

$$M = \int_a^\xi (\xi - x)\, d\varphi(x), \tag{6.66}$$

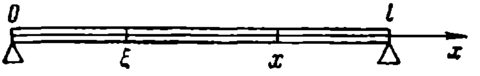

FIG. 30

where $\varphi(x)$ denotes the shearing stress in the section x, i.e. the sum of all forces acting in the interval $[0, x]$, both the distributed ones and the concentrated ones, including the reactions of the supports.

§ 4. Integrals and Derivatives of Fractional Orders

1. In operational calculus and in some other branches of mathematics one sometimes encounters the concept of integrals and derivatives of fractional orders. The simplest and most natural definition can be given with the aid of Cauchy formula for the nth original which was given in Chapter V, § 1.

Indeed, after dropping from this formula the polynomial $P_{n-1}(x)$ with arbitrary coefficients it can be supposed, by definition, that the n-fold integral of the function $f(x)$ equals

$$\underbrace{\int_a^x dx \int_a^x dx \ldots \int_a^x}_{n \text{ times}} f(x)\, dx = \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1} f(t)\, dt. \tag{6.67}$$

The latter equation can be considered for *any* value of n, if the factorial is exchanged for a gamma-function: $(n - 1)! = \Gamma(n)$. Then for any n the expression of the form

$$\frac{1}{\Gamma(n)} \int_a^x (x - t)^{n-1} f(t)\, dt, \tag{6.68}$$

is called *the integral of the n-th order of the function* $f(x)$.

Theorems relating to the existence of such integrals can be obtained from the theorems of Chapter V, §§ 1 and 3. Thus, *if the function $f(x)$ is continuous in the interval $[a, b]$, then for all $n > 1$ there exists its integral of the n-th order in this interval.*

2. The possibility of differentiating integrals is of great importance. If the function $f(x)$ has an integral of the nth order, then it has also an integral of the order $n + 1$. The latter is a *differentiable function, whose derivative equals the integral of the n-th order.*

The latter proposition also enables us to define the derivatives of any order k, including a fractional one. In order to obtain a derivative of the order $k > 0$, it is necessary to find the integral of the order $1 - \{k\}$ of the function $f(x)$, and then, from the integral obtained, the derivative of the order $[k] + 1$ in the usual sense ($[k]$ denotes the integral part of the number k, and $\{k\} = k - [k]$ its fractional part). It is clear that the orders of the integral and of the derivative may be increased simultaneously by any integer.

For the existence of a derivative of a fractional order $k > 0$ it is sufficient to assume the existence of the ordinary derivative of the $([k] + 1)$th order of the function $f(x)$. Assuming that the derivatives and integrals obtained exist, the operations of differentiation and integration of fractional orders are commutative.

EXAMPLE 15. Let the function $f(x) = x$ be defined in the interval [0, 1]. Since it becomes zero at the left end of the interval, there exists for it an integral of any order $n > 0$. For example, for $n = \frac{1}{2}$ we obtain an integral equal

$$\frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{-\frac{1}{2}}\, t\, dt = \frac{1}{\sqrt{\pi}}\int_0^x \frac{t\, dt}{\sqrt{x-t}} = \frac{4}{3\sqrt{\pi}}\, x^{\frac{3}{2}}.$$

Differentiating the integral of the order $\frac{1}{2}$ we obtain a derivative of order $\frac{1}{2}$ of the function $f(x) = x$, equal

$$f^{(\frac{1}{2})}(x) = \frac{2}{\sqrt{\pi}}\, x^{\frac{1}{2}}.$$

CHAPTER VII

THE TRANSFORMATION OF DIFFERENTIAL AND INTEGRAL EXPRESSIONS

§ 1. The Transformation of Differential Expressions

The Linear Case

1. Suppose we are given *the differential expression*

$$V = F(x, y, y', y'', \ldots), \tag{7.1}$$

in which the function $y = f(x)$ and its derivatives $y', y'', \ldots$, are defined in some interval I of the axis Ox. In transforming V to some other variables it is necessary to effect a change, (a) of the independent variable x, (b) of the function y, (c) of both the independent variable x and the function y. In all these cases, it is very difficult to obtain a formula for the expression $y^{(n)}_{x^n}$ by means of new variables. Therefore it is necessary to find $y'_{x^1}, y''_{x^2}, y'''_{x^3}$, consecutively with the aid of rules of differentiation.

(a) Let

$$x = \varphi(u), \tag{7.2}$$

where the function $\varphi(u)$ is continuous and so are its derivatives up to the order required, and $x' = \varphi'(u) \neq 0$. Then (in the right-hand sides of the following equations the derivatives with respect to u are denoted by strokes)

$$\left.\begin{aligned} y'_x &= \frac{y'}{x'}, \\ y''_{x^2} &= \frac{y''x' - x''y'}{x'^3}, \\ y'''_{x^3} &= \frac{y'''x'^2 - x'''y'x' - 3y''x''x' + 3x''^2y'}{x'^5}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{7.3}$$

Having found in succession the expressions for the derivatives of all orders required, we have the opportunity to construct the required new expression for V:

$$V = F\left[\varphi(u), f(\varphi(u)), \frac{y'}{x'}, \frac{y''x' - x''y'}{x'^3}, \ldots\right] = F_1(u, y, y', y'', \ldots). \tag{7.4}$$

(b) Let

$$y = \psi(v), \tag{7.5}$$

where functions $\psi(v)$ and $v(x)$ are continuous with respect to their arguments as are the derivatives up to the required order. Then

$$\left.\begin{aligned} y'_x &= y'_v v', \\ y''_{x^2} &= y''_{v^2} v'^2 + y'_v v'', \\ y'''_{x^3} &= y'''_v v'^3 + 3y''_{v^2} v' v'' + y'_v v''', \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{7.6}$$

Having found the derivatives of all orders, we are able to construct the required new expression for V:

$$V = F[x, \psi(v), \psi'_v v', \psi''_{v^2} v'^2 + \psi'_v v'', \ldots] = F_2(x, v, v', v'', \ldots). \tag{7.7}$$

(c) Let

$$x = \varphi(u, v), \quad y = \psi(u, v), \tag{7.8}$$

where function $\varphi(u, v)$ and $\psi(u, v)$ are continuous, and so are their derivatives up to the order required. Regarding v as a new function of a new independent variable u, we find

$$\left.\begin{aligned} y'_x &= \frac{\dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v} v'}{\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v} v'}, \\ y''_{x^2} &= \frac{\left(\dfrac{\partial^2 y}{\partial u^2} + 2\dfrac{\partial^2 y}{\partial u\,\partial v} v' + \dfrac{\partial^2 y}{\partial v^2} v'^2 + \dfrac{\partial y}{\partial v} v''\right)\left(\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v} v'\right)}{\left(\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v} v'\right)^3} \\ &\quad - \frac{\left(\dfrac{\partial^2 x}{\partial u^2} + 2\dfrac{\partial^2 x}{\partial u\,\partial v} v' + \dfrac{\partial^2 x}{\partial v^2} v'^2 + \dfrac{\partial x}{\partial v} v''\right)\left(\dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v} v'\right)}{\left(\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v} v'\right)^2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{7.9}$$

Continuing the differentiation further, we are enabled to construct the required new expression for V:

$$\begin{aligned} V &= F\left[\varphi(u, v), \psi(u, v), \frac{y'_u + y'_v v'}{x'_u + x'_v v'}, \ldots\right] \\ &= F_3(u, v, v', v'', \ldots). \end{aligned} \tag{7.10}$$

The Plane Case

2. Suppose we are given *the differential expression*

$$W = F(x, y, z, z'_x, z'_y, z''_{x^2}, z''_{xy}, z''_{y^2}, \ldots), \tag{7.11}$$

in which x and y are independent variables, z is a function of x and y, which is defined, with its derivatives, in some region D of the plane Oxy. In the transformation of W, cases are encountered in which it is necessary to change (a) the independent variables x and y, (b) the function z, (c) both the independent variables x, y and the function z. Here each time one has to carry out the calculations consecutively, with the aid of rules of differentiation, since there are difficulties in obtaining any kind of general formulae.

(a) Let

$$x = \varphi(u, v), \quad y = \psi(u, v), \tag{7.12}$$

where the functions $\varphi(u, v)$ and $\psi(u, v)$ together with their derivatives up to the order required are continuous, and the Jacobian $\partial(x, y)/\partial(u, v) \neq 0$. Then,

$$\left.\begin{aligned} z'_x &= \frac{\dfrac{\partial(z, y)}{\partial(u, v)}}{\dfrac{\partial(x, y)}{\partial(u, v)}}, \\[2ex] z'_y &= \frac{\dfrac{\partial(x, z)}{\partial(u, v)}}{\dfrac{\partial(x, y)}{\partial(x, v)}}. \end{aligned}\right\} \tag{7.13}$$

In order to obtain expressions of higher derivatives of z in terms of u and v, the given equations can be differentiated with respect to x and y, changing $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ each time,

according to the following formulae:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\dfrac{\partial y}{\partial v}}{\dfrac{\partial(x, y)}{\partial(u, v)}}, \quad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial x}{\partial v}}{\dfrac{\partial(x, y)}{\partial(u, v)}}, \\ \frac{\partial v}{\partial x} &= -\frac{\dfrac{\partial y}{\partial u}}{\dfrac{\partial(x, y)}{\partial(u, v)}}, \quad \frac{\partial v}{\partial y} = \frac{\dfrac{\partial v}{\partial u}}{\dfrac{\partial(x, y)}{\partial(u, x)}}. \end{aligned}\right\} \tag{7.14}$$

Having transformed the derivatives of all orders required we are able to construct the required new expression for W:

$$W = F[\varphi(u, v), \psi(u, v), z, z'_x, z'_y, \dots] = F_1(u, v, z, z'_u, z'_v, \dots). \tag{7.15}$$

(b) Suppose

$$z = \psi(w), \tag{7.16}$$

where $\psi(w)$ and $w(x, y)$ and their derivatives up to the order required are continuous. Then,

$$\left.\begin{aligned} z'_x &= \psi'(w)w'_x, \quad z'_y = \psi'(w)w'_y, \\ z''_{x^2} &= \psi''(w)w'^2_x + \psi'(w)w''_{x^2}, \\ z''_{xy} &= \psi''(w)w'_x w'_y + \psi'(w)w''_{xy}, \\ z''_{y^2} &= \psi''(w)w'^2_y + \psi'(w)\,(w)''_{y^2}, \\ & \dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{7.17}$$

Continuing to differentiate consecutively, we are able to construct the required new expression for W:

$$\begin{aligned} W &= F[(x, y, \psi(w), \psi'(w)w'_x, \psi'(w)w'_y, \dots] \\ &= F_2(x, y, w, w'_x, w'_y, \dots). \end{aligned} \tag{7.18}$$

(c) Let

$$x = \varphi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w), \tag{7.19}$$

where the functions φ, ψ, χ and their derivatives up to the order required are continuous. Differentiating the third of the above

equations with respect to x and y, we get:

$$\left.\begin{aligned} z'_x &= \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial z}{\partial v} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x}, \\ z'_y &= \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial z}{\partial v} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y}. \end{aligned}\right\} \tag{7.20}$$

In these equations, it is still necessary to express $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ in terms of new variables, which can be found from systems obtained by differentiation of the first two of the selected equations with respect to x and y:

$$\left.\begin{aligned} 1 &= \left(\frac{\partial x}{\delta u} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial x}{\partial v} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x}, \\ 0 &= \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial y}{\partial v} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x}; \end{aligned}\right\} \tag{7.21}$$

$$\left.\begin{aligned} 0 &= \left(\frac{\partial x}{\partial u} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial x}{\partial v} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y}, \\ 1 &= \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial y}{\partial v} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y}. \end{aligned}\right\} \tag{7.22}$$

This method can also be used to obtain expressions for the derivatives of the succeeding orders in terms of new variables. This provides the possibility of constructing a new expression for W:

$$\begin{aligned} W &= F[\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w), z'_x, z'_y, \ldots] \\ &= F_3(u, v, w, w'_u, w'_v, \ldots). \end{aligned} \tag{7.23}$$

The Space Case

3. Suppose we are given *the differential expression*

$$W = F(x, y, z, t, t'_x, t'_y, t'_z, \ldots), \tag{7.24}$$

in which x, y, z are independent variables and t is a function of x, y, z defined, together with its derivatives up to the required order, in some region V or space $Oxyz$.

If it is necessary to transform W to new variables u, v, w, connected with x, y, z by the relationship

$$x = \varphi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w), \tag{7.25}$$

the following formulae can be used

$$t'_x = \frac{\dfrac{\partial(t, y, z)}{\partial(u, v, w)}}{\dfrac{\partial(x, y, z)}{\partial(u, v, w)}}, \quad t'_y = \frac{\dfrac{\partial(x, t, z)}{\partial(u, v, w)}}{\dfrac{\partial(x, y, z)}{\partial(u, v, w)}}, \quad t'_z = \frac{\dfrac{\partial(x, y, t)}{\partial(u, v, w)}}{\dfrac{\partial(x, y, z)}{\partial(u, v, w)}}. \tag{7.26}$$

It is possible to obtain expressions for derivatives in t of higher orders by a consecutive differentiation of the given equations with respect to x, y and z, each time changing u'_x, v'_x, w'_x in the result according to the formulae

$$u'_x = \frac{\dfrac{\partial(y, z)}{\partial(v, w)}}{\dfrac{\partial(x, y, z)}{\partial(u, v, w)}}, \quad v'_x = -\frac{\dfrac{\partial(y, z)}{\partial(u, w)}}{\dfrac{\partial(x, y, z)}{\partial(u, v, w)}}, \quad w'_x = \frac{\dfrac{\partial(y, z)}{\partial(u, v)}}{\dfrac{\partial(x, y, z)}{\partial(u, v, w)}}, \tag{7.27}$$

and u'_y, v'_y, w'_y and u'_z, v'_z, w'_z, according to analogous formulae. This enables us to construct the required new expressions for W

$$\begin{aligned} W &= F[\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w), t, t'_x, t'_y, t'_z, \ldots] \\ &= F_1(u, v, w, t, t'_u, t'_v, t'_w, \ldots). \end{aligned} \tag{7.28}$$

4. Expressions are given below for the principal differential operators (gradient, divergence, curl, Laplacian). These expressions are obtained by transforming rectangular cartesian coordinates to various curvilinear orthogonal coordinates.

1°. *The plane case.* Let $z = Z(x, y)$ be a twice differentiable function of x and y, and let $\mathbf{a} = \mathbf{a}(x, y)$ be a differentiable vector function of x and y. The principal differential operators are defined by the following relationships:

$$\text{gradient:} \quad \operatorname{grad} z = \left\{\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right\}; \tag{7.29}$$

$$\text{divergence:} \quad \operatorname{div} \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y}; \tag{7.30}$$

$$\text{Laplacian:} \quad \Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}. \tag{7.31}$$

In transforming to new independent variables u and v (on condition that they constitute a system of curvilinear orthogonal coordinates, see Chapter IV, § 2, sec. 5),

$$x = \varphi(u, v), \quad y = \psi(u, v), \tag{7.32}$$

it is possible to obtain the following expressions for these differential operations

$$\operatorname{grad}_u z = \frac{1}{l_u} \frac{\partial z}{\partial u}, \quad \operatorname{grad}_v z = \frac{1}{l_v} \frac{\partial z}{\partial v}, \tag{7.33}$$

$$\operatorname{div} \mathbf{a} = \frac{1}{l_u l_v} \left[\frac{\partial}{\partial u} (l_v a_u) + \frac{\partial}{\partial v} (l_u a_v) \right], \tag{7.34}$$

$$\Delta z = \frac{1}{l_u l_v} \left[\frac{\partial}{\partial u} \left(\frac{l_v}{l_u} \frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{l_u}{l_v} \frac{\partial z}{\partial v} \right) \right], \tag{7.35}$$

where l_u and l_v are the corresponding Lamé coefficients. Hence we have:

(a) in polar coordinates (Chapter IV, § 2, sec. 5, 2°):

$$\operatorname{grad}_u z = \frac{\partial z}{\partial u}, \quad \operatorname{grad}_v z = \frac{1}{u} \frac{\partial z}{\partial v}, \tag{7.36}$$

$$\operatorname{div} \boldsymbol{a} = \frac{1}{u} a_u + \frac{\partial a_u}{\partial u} + \frac{1}{u} \frac{\partial a_v}{\partial v}, \tag{7.37}$$

$$\Delta z = \frac{\partial^2 z}{\partial u^2} + \frac{1}{u} \frac{\partial z}{\partial u} + \frac{1}{u^2} \frac{\partial^2 z}{\partial v^2}; \tag{7.38}$$

(b) in degenerate elliptical coordinates (Chapter IV, § 2, sec. 5, 5°):

$$\operatorname{grad}_u z = \frac{1}{\sqrt{\cosh^2 u - \cos^2 v}} \frac{\partial z}{\partial u},$$

$$\operatorname{grad}_v z = \frac{1}{\sqrt{\cosh^2 u - \cos^2 v}} \frac{\partial z}{\partial v}, \tag{7.39}$$

$$\operatorname{div} \mathbf{a} = \frac{1}{\sqrt{\cosh^2 u - \cos^2 v}} \left(\frac{\partial a_u}{\partial u} + \frac{\partial a_v}{\partial v} \right)$$

$$+ \frac{1}{\sqrt{(\cosh^2 u - \cos^2 v)^3}} (a_u \cosh u \sinh u + a_v \cos v \sin v), \tag{7.40}$$

$$\Delta z = \frac{1}{\cosh^2 u - \cos^2 v} \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right); \tag{7.41}$$

(c) in parabolic coordinates (Chapter IV, § 2, sec. 5, 6°):

$$\operatorname{grad}_u z = \frac{1}{2\sqrt{u^2 + v^2}} \frac{\partial z}{\partial u}, \quad \operatorname{grad}_v z = \frac{1}{2\sqrt{u^2 + v^2}} \frac{\partial z}{\partial v}, \tag{7.42}$$

$$\operatorname{div} \mathbf{a} = \frac{1}{2\sqrt{u^2 + v^2}} \left(\frac{\partial a_u}{\partial u} + \frac{\partial a_v}{\partial v} \right)$$

$$+ \frac{1}{2\sqrt{(u^2 + v^2)^3}} (u a_u + v a_v), \tag{7.43}$$

$$\Delta z = \frac{1}{4(u^2 + v^2)} \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right); \tag{7.44}$$

(d) in bipolar coordinates (Chapter IV, § 2, sec. 5, 7°):

$$\operatorname{grad}_u z = (\cosh u + \cos v) \frac{\partial z}{\partial u}, \quad \operatorname{grad}_v z = (\cosh u + \cos v) \frac{\partial z}{\partial v}, \tag{7.45}$$

$$\operatorname{div} \mathbf{a} = (\cosh u + \cos v) \left(\frac{\partial a_u}{\partial u} + \frac{\partial a_v}{\partial v} \right) + a_v \sin v - a_u \sinh u, \tag{7.46}$$

$$\Delta z = (\cosh u + \cos v)^2 \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right). \tag{7.47}$$

2°. *The space case.* Let $t = t(x, y, z)$ be a twice differentiable function of x, y, and z and let $\mathbf{a} = \mathbf{a}(x, y, z)$ be a differentiable vector function of x, y and z. The principal differential operators are

defined by the following relationships:

$$\text{gradient:}\quad \operatorname{grad} t = \left\{\frac{\partial t}{\partial x},\ \frac{\partial t}{\partial y},\ \frac{\partial t}{\partial z}\right\}, \tag{7.48}$$

$$\text{divergence:}\quad \operatorname{div} \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}, \tag{7.49}$$

$$\operatorname{rot} \boldsymbol{a} = \left\{\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z},\ \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x},\ \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right\}, \tag{7.50}$$

$$\text{Laplacian:}\quad \Delta t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}. \tag{7.51}$$

On transforming to new independent variables u, v and w (on condition that they constitute a system of curvilinear orthogonal coordinates, see Chapter IV, § 3, sec. 2),

$$x = \varphi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w), \tag{7.52}$$

we can obtain the following expressions for these differential operators:

$$\operatorname{grad}_u t = \frac{1}{L_u}\frac{\partial t}{\partial u}, \quad \operatorname{grad}_v t = \frac{1}{L_v}\frac{\partial t}{\partial v}, \quad \operatorname{grad}_w t = \frac{1}{L_w}\frac{\partial t}{\partial w}, \tag{7.53}$$

$$\operatorname{div} \mathbf{a} = \frac{1}{L_u L_v L_w}\left[\frac{\partial}{\partial u}(L_v L_w a_u) + \frac{\partial}{\partial v}(L_w L_u a_v) + \frac{\partial}{\partial w}(L_u L_v a_w)\right], \tag{7.54}$$

$$\left.\begin{aligned}
\operatorname{rot}_u \mathbf{a} &= \frac{1}{L_v L_w}\left[\frac{\partial}{\partial v}(L_w a_w) - \frac{\partial}{\partial w}(L_v a_v)\right],\\
\operatorname{rot}_v \mathbf{a} &= \frac{1}{L_w L_u}\left[\frac{\partial}{\partial w}(L_u a_u) - \frac{\partial}{\partial u}(L_w a_w)\right],\\
\operatorname{rot}_w \mathbf{a} &= \frac{1}{L_u L_v}\left[\frac{\partial}{\partial u}(L_v a_v) - \frac{\partial}{\partial v}(L_u a_u)\right],
\end{aligned}\right\} \tag{7.55}$$

$$\begin{aligned}
\Delta t = \frac{1}{L_u L_v L_w}\Bigg[&\frac{\partial}{\partial u}\left(\frac{L_v L_w}{L_u}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{L_w L_u}{L_v}\frac{\partial t}{\partial v}\right)\\
&+ \frac{\partial}{\partial w}\left(\frac{L_u L_v}{L_w}\frac{\partial t}{\partial w}\right)\Bigg],
\end{aligned} \tag{7.56}$$

where L_u, L_v, L_w are the corresponding Lamé coefficients. Hence we have:

(a) in cylindrical coordinates (Chapter IV, § 3, sec. 2, 2°):

$$\operatorname{grad}_u t = \frac{\partial t}{\partial u}, \quad \operatorname{grad}_v t = \frac{1}{u}\frac{\partial t}{\partial v}, \quad \operatorname{grad}_w t = \frac{\partial t}{\partial w}, \tag{7.57}$$

$$\operatorname{div} \mathbf{a} = \frac{1}{u} a_u + \frac{\partial a_u}{\partial u} + \frac{1}{u}\frac{\partial a_v}{\partial v} + \frac{\partial a_w}{\partial w}, \tag{7.58}$$

$$\left.\begin{aligned} \operatorname{rot}_u \mathbf{a} &= \frac{1}{u}\frac{\partial a_w}{\partial v} - \frac{\partial a_v}{\partial w}, \\ \operatorname{rot}_v \mathbf{a} &= \frac{\partial a_u}{\partial w} - \frac{\partial a_w}{\partial u}, \\ \operatorname{rot}_w \mathbf{a} &= \frac{1}{u} a_v + \frac{\partial a_v}{\partial u} - \frac{1}{u}\frac{\partial a_u}{\partial v}, \end{aligned}\right\} \tag{7.59}$$

$$\Delta t = \frac{1}{u}\frac{\partial t}{\partial u} + \frac{\partial^2 t}{\partial u^2} + \frac{1}{u^2}\frac{\partial^2 t}{\partial v^2} + \frac{\partial^2 t}{\partial w^2}; \tag{7.60}$$

(b) in spherical coordinates (Chapter IV, § 3, sec. 2, 3°):

$$\operatorname{grad}_u t = \frac{\partial t}{\partial u}, \quad \operatorname{grad}_v t = \frac{1}{u \sin w}\frac{\partial t}{\partial v}, \quad \operatorname{grad}_w t = \frac{1}{u}\frac{\partial t}{\partial w}, \tag{7.61}$$

$$\operatorname{div} \mathbf{a} = \frac{2}{u} a_u + \frac{1}{u \tan w} a_w + \frac{\partial a_u}{\partial u} + \frac{1}{u \sin w}\frac{\partial a_v}{\partial v} + \frac{1}{n}\frac{\partial a_w}{\partial w}, \tag{7.62}$$

$$\left.\begin{aligned} \operatorname{rot}_u \mathbf{a} &= \frac{1}{u \sin w}\frac{\partial a_w}{\partial v} - \frac{1}{u}\frac{\partial a_v}{\partial w} - \frac{1}{u \tan w} a_v, \\ \operatorname{rot}_v \mathbf{a} &= \frac{1}{u}\frac{\partial a_u}{\partial w} - \frac{\partial a_w}{\partial u} - \frac{a_w}{u}, \\ \operatorname{rot}_w \mathbf{a} &= \frac{\partial a_v}{\partial u} + \frac{1}{u} a_v - \frac{1}{u \sin w}\frac{\partial a_u}{\partial v}, \end{aligned}\right\} \tag{7.63}$$

$$\Delta t = \frac{2}{u}\frac{\partial t}{\partial u} + \frac{\partial^2 t}{\partial u^2} + \frac{1}{u^2 \sin^2 w}\frac{\partial^2 t}{\partial v^2} + \frac{1}{u^2 \tan w}\frac{\partial t}{\partial w} + \frac{1}{u^2}\frac{\partial^2 t}{\partial w^2}; \tag{7.64}$$

(c) in degenerate ellipsoidal "elongated" coordinates (Chapter IV, § 3, sec. 2, 5⁰):

$$\left.\begin{aligned} \operatorname{grad}_u t &= \frac{1}{\sqrt{\sinh^2 u + \sin^2 w}}\frac{\partial t}{\partial u},\\ \operatorname{grad}_v t &= \frac{1}{\sinh u \sin w}\frac{\partial t}{\partial v},\\ \operatorname{grad}_w t &= \frac{1}{\sqrt{\sinh^2 u + \sin^2 w}}\frac{\partial t}{\partial w}, \end{aligned}\right\} \tag{7.65}$$

$$\begin{aligned} \operatorname{div} \mathbf{a} &= \frac{1}{\sqrt{(\sinh^2 u + \sin^2 w)^3}} \times\\ &\times \left(\frac{2\sinh^2 u + \sin^2 w}{\tanh u} a_u + \frac{2\sin^2 w + \sinh^2 u}{\tan w} a_w\right)\\ &+ \frac{1}{\sqrt{\sinh^2 u + \sin^2 w}}\left(\frac{\partial a_u}{\partial u} + \frac{\partial a_w}{\partial w}\right) + \frac{1}{\sinh u \sin w}\frac{\partial a_v}{\partial v}, \end{aligned} \tag{7.66}$$

$$\left.\begin{aligned} \operatorname{rot}_u \mathbf{a} &= \frac{1}{\sinh u \sin w}\frac{\partial a_w}{\partial v}\\ &\quad - \frac{1}{\sqrt{\sinh^2 u + \sin^2 w}}\left(\frac{\partial a_v}{\partial w} + \frac{1}{\tan w} a_v\right),\\ \operatorname{rot}_v \mathbf{a} &= \frac{1}{\sqrt{\sinh^2 u + \sin^2 w)^3}}(a_u \sin w \cos w - a_w \tan u \cosh u)\\ &\quad + \frac{1}{\sqrt{\sinh^2 u + \sin^2 w}}\left(\frac{\partial a_u}{\partial w} - \frac{\partial a_w}{\partial u}\right),\\ \operatorname{rot}_w \mathbf{a} &= \frac{1}{\sqrt{\sinh^2 u + \sin^2 w}}\left(\frac{1}{\tanh u} a_v + \frac{\partial a_v}{\partial u}\right)\\ &\quad - \frac{1}{\sinh u \sin w}\frac{\partial a_u}{\partial v}, \end{aligned}\right\} \tag{7.67}$$

$$\Delta t = \frac{1}{\sinh^2 u + \sin^2 w}\left[\frac{\partial^2 t}{\partial u^2} + \frac{1}{\tanh u}\frac{\partial t}{\partial u} + \frac{\partial^2 t}{\partial w^2}\right.$$
$$\left. + \frac{1}{\tan w}\frac{\partial t}{\partial w} + \left(\frac{1}{\sin^2 w} + \frac{1}{\sinh^2 u}\right)\frac{\partial^2 t}{\partial v^2}\right]; \qquad (7.68)$$

(d) in degenerate ellipsoidal "flattened" coordinates (Chapter IV, § 3, sec. 2, 6⁰):

$$\left.\begin{aligned} \operatorname{grad}_u t &= \frac{1}{\sqrt{\sinh^2 u + \cos^2 w}}\frac{\partial t}{\partial u}, \\ \operatorname{grad}_v t &= \frac{1}{\cosh u \sin w}\frac{\partial t}{\partial v}, \\ \operatorname{grad}_w t &= \frac{1}{\sqrt{\sinh^2 u + \cos^2 w}}\frac{\partial t}{\partial w}, \end{aligned}\right\} \qquad (7.69)$$

$$\begin{aligned} \operatorname{div} \mathbf{a} &= \frac{1}{\sqrt{\sinh^2 u + \cos^2 w)^3}} \times \\ &\times [(\sinh^2 u + \cosh^2 u + \cos^2 w)a_u \tanh u \\ &+ (\sinh^2 u + \cos^2 w - \sin^2 w)a_w \cot w] \\ &+ \frac{1}{\sqrt{\sinh^2 u + \cos^2 w}}\left(\frac{\partial a_u}{\partial u} + \frac{\partial a_w}{\partial w}\right) + \frac{1}{\cosh u \sin w}\frac{\partial a_v}{\partial u}, \end{aligned} \qquad (7.70)$$

$$\left.\begin{aligned} \operatorname{rot}_u \mathbf{a} &= \frac{1}{\cosh u \sin w}\frac{\partial a_w}{\partial v} \\ &\quad - \frac{1}{\sqrt{\sinh^2 u + \cos^2 w}}\left(\frac{\partial a_v}{\partial w} + \frac{1}{\tan w}a_v\right), \\ \operatorname{rot}_v \mathbf{a} &= \frac{1}{\sqrt{\sinh^2 u + \cos^2 w}}\left(\frac{\partial a_u}{\partial w} - \frac{\partial a_w}{\partial u}\right) \\ &- \frac{1}{\sqrt{\sinh^2 u + \cos^2 w)^3}}(a_u \sin w \cos w + a_w \sinh u \cosh u), \\ \operatorname{rot}_w \mathbf{a} &= \frac{1}{\sqrt{\sinh^2 u + \cos^2 w}}\left(\frac{\partial a_v}{\partial u} + a_v \tanh u\right) \\ &\quad - \frac{1}{\cosh u \sin w}\frac{\partial a_u}{\partial v}, \end{aligned}\right\} \qquad (7.71)$$

$$\Delta t = \frac{1}{\sinh^2 u + \cos^2 w}\left[\frac{\partial^2 t}{\partial u^2} + \tanh u \frac{\partial t}{\partial u} + \frac{\partial^2 t}{\partial w^2} + \cot w \frac{\partial t}{\partial w} + \left(\frac{1}{\sin^2 w} - \frac{1}{\cosh^2 u}\right)\frac{\partial^2 t}{\partial v^2}\right]; \tag{7.72}$$

(e) in parabolical coordinates (Chapter IV, § 3, sec. 2, 8⁰):

$$\left.\begin{aligned} \operatorname{grad}_u t &= \frac{1}{2\sqrt{u^2 + w^2}}\frac{\partial t}{\partial u}, \\ \operatorname{grad}_v t &= \frac{1}{2uw}\frac{\partial t}{\partial v}, \\ \operatorname{grad}_w t &= \frac{1}{2\sqrt{u^2 + w^2}}\frac{\partial t}{\partial w}, \end{aligned}\right\} \tag{7.73}$$

$$\operatorname{div} \mathbf{a} = \frac{1}{2uw\sqrt{(u^2 + w^2)^3}}\left[wa_u(2u^2 + w^2) + ua_w(u^2 + 2w^2)\right] + \frac{1}{2\sqrt{u^2 + w^2}}\left(\frac{\partial a_u}{\partial u} + \frac{\partial a_w}{\partial w}\right) + \frac{1}{2uw}\frac{\partial a_v}{\partial v}, \tag{7.74}$$

$$\left.\begin{aligned} \operatorname{rot}_u \mathbf{a} &= \frac{1}{2uw}\frac{\partial a_w}{\partial v} - \frac{1}{2w\sqrt{u^2 + w^2}}\left(a_v + w\frac{\partial a_v}{\partial w}\right), \\ \operatorname{rot}_v \mathbf{a} &= \frac{1}{2\sqrt{(u^2 + w^2)^3}}(wa_u - ua_w) \\ &\quad + \frac{1}{2\sqrt{u^2 + w^2}}\left(\frac{\partial a_u}{\partial w} - \frac{\partial a_w}{\partial u}\right), \\ \operatorname{rot}_w \mathbf{a} &= \frac{1}{2u\sqrt{u^2 + w^2}}\left(a_v + u\frac{\partial a_v}{\partial u}\right) - \frac{1}{2uw}\frac{\partial a_u}{\partial v}, \end{aligned}\right\} \tag{7.75}$$

$$\Delta t = \frac{1}{4uw(u^2 + w^2)}\left(u\frac{\partial t}{\partial w} + w\frac{\partial t}{\partial u}\right) + \frac{1}{4(u^2 + w^2)}\left(\frac{\partial^2 t}{\partial u^2} + \frac{\partial^2 t}{\partial w^2}\right) + \frac{1}{4u^2w^2}\frac{\partial^2 t}{dv^2}; \tag{7.76}$$

(f) in toroidal coordinates (Chapter IV, § 3, sec. 2, 9°):

$$\left.\begin{aligned}
\operatorname{grad}_u t &= (\cosh u - \cos w)\frac{\partial t}{\partial u}, \\
\operatorname{grad}_v t &= \frac{\cosh u - \cos w}{\sinh u}\frac{\partial t}{\partial v}, \\
\operatorname{grad}_w t &= (\cosh u - \cos w)\frac{\partial t}{\partial w},
\end{aligned}\right\} \tag{7.77}$$

$$\begin{aligned}
\operatorname{div} \mathbf{a} &= \mathbf{a}_u \frac{1 - \sinh^2 u - \cosh u \cos w}{\sinh u} - 2a_w \sin w \\
&+ (\cosh u - \cos w)\left(\frac{\partial a_u}{\partial u} + \frac{1}{\sinh u}\frac{\partial a_v}{\partial v} + \frac{\partial a_w}{\partial w}\right),
\end{aligned} \tag{7.78}$$

$$\left.\begin{aligned}
\operatorname{rot}_u \mathbf{a} &= (\cosh u - \cos w)\left(\frac{1}{\sinh u}\frac{\partial a_w}{\partial v} - \frac{\partial a_v}{\partial w}\right) + a_v \sin w, \\
\operatorname{rot}_v \mathbf{a} &= (\cosh u - \cos w)\left(\frac{\partial a_u}{\partial w} - \frac{\partial a_w}{\partial u}\right) - a_u \sin w + a_w \sinh u, \\
\operatorname{rot}_w \mathbf{a} &= (\cosh u - \cos w)\left(\frac{\partial a_v}{\partial u} - \frac{1}{\sinh u}\frac{\partial a_u}{\partial v}\right) \\
&+ a_v \frac{1 - \cosh u \cos w}{\sinh u},
\end{aligned}\right\} \tag{7.79}$$

$$\begin{aligned}
\Delta t &= (\cosh u - \cos w)\left(\frac{(1 - \cosh u \cos w)}{\sinh u}\frac{\partial t}{\partial u} - \sin w \frac{\partial t}{\partial w}\right) \\
&+ (\cosh u - \cos w)^2\left(\frac{\partial^2 t}{\partial u^2} + \frac{1}{\sinh^2 u}\frac{\partial^2 t}{\partial v^2} + \frac{\partial^2 t}{\partial w^2}\right);
\end{aligned} \tag{7.80}$$

(g) in bipolar coordinates (Chapter IV, § 3, sec. 2, 10°):

$$\left.\begin{aligned}
\operatorname{grad}_u t &= (\cosh w - \cos u)\frac{\partial t}{\partial u},\\
\operatorname{grad}_v t &= \frac{\cosh w - \cos u}{\sin u}\frac{\partial t}{\partial v},\\
\operatorname{grad}_w t &= (\cosh w - \cos u)\frac{\partial t}{\partial w},
\end{aligned}\right\} \tag{7.81}$$

$$\begin{aligned}
\operatorname{div}\mathbf{a} &= a_u\frac{\cos u\cosh w - \sin^2 u - 1}{\sin u} - 2a_w\sinh w\\
&\quad + (\cosh w - \cos u)\left(\frac{\partial a_u}{\partial u} + \frac{1}{\sin u}\frac{\partial a_v}{\partial v} + \frac{\partial a_w}{\partial w}\right),
\end{aligned} \tag{7.82}$$

$$\left.\begin{aligned}
\operatorname{rot}_u\mathbf{a} &= (\cosh w - \cos u)\left(\frac{1}{\sin u}\frac{\partial a_w}{\partial v} - \frac{\partial a_v}{\partial w}\right) - a_v\sinh w,\\
\operatorname{rot}_v\mathbf{a} &= (\cosh w - \cos u)\left(\frac{\partial a_u}{\partial w} - \frac{\partial a_w}{\partial u}\right) - a_u\sinh w + a_w\sin u,\\
\operatorname{rot}_w\mathbf{a} &= (\cosh w - \cos u)\left(\frac{\partial a_v}{\partial u} - \frac{1}{\sin u}\frac{\partial a_u}{\partial v}\right)\\
&\quad + a_v\frac{\cos u\cosh w - 1}{\sin u},
\end{aligned}\right\} \tag{7.83}$$

$$\begin{aligned}
\Delta t &= (\cosh w - \cos u)\left(\frac{\cos u\cosh w - 1}{\sin u}\frac{\partial t}{\partial u} - \sinh w\frac{\partial t}{\partial w}\right)\\
&\quad + (\cosh w - \cos u)^2\left(\frac{\partial^2 t}{\partial u^2} + \frac{1}{\sin^2 u}\frac{\partial^2 t}{\partial v^2} + \frac{\partial^2 t}{\partial w^2}\right).
\end{aligned} \tag{7.84}$$

§ 2. The Transformation of Integral Expressions

The Integral over the Measure of a Region

1. Let $f(P)$ be a function of a point, defined in the region E. The integral over the measure of a region (see Chapter V, § 4) possesses the following properties:

(a) *Linearity:*

$$\int_E [C_1 f_1(P) + C_2 f_2(P) + \cdots + C_k f_k(P)]\, de$$
$$= C_1 \int_E f_1(P)\, de + C_2 \int_E f_2(P)\, de + \cdots + C_k \int_E f_k(P)\, de, \tag{7.85}$$

where $C_1, C_2, \ldots, C_k$ are constants.

(b) *Additivity:*

$$\int_{E_1+E_2+\cdots+E_k} f(P)\, de = \int_{E_1} f(P)\, de + \int_{E_2} f(P)\, de + \cdots + \int_{E_k} f(P)\, de, \tag{7.86}$$

where $E_1 + E_2 + \cdots + E_k$ is understood as the set of all regions $E_1, E_2, \ldots, E_k$.

(c) *Boundedness:*

$$m\Delta E \leqslant \int_E f(P)\, de \leqslant M\Delta E, \tag{7.87}$$

where ΔE is the measure of the region E, and m and M are the least and the greatest values of the function $f(P)$ in the region E respectively. The given inequality expresses *the theorem for the estimate of an integral*; from it there follows *the mean value theorem:*

$$\int_E f(P)\, de = f(P_c)\Delta E, \tag{7.88}$$

where P_c is some interior point of the region E.

(d) *Differentiability* over the measure of the region:

$$\frac{d}{de}\int_e f(P)\, de = f(P), \quad \text{or} \quad d\int_e f(P)\, de = f(P)\, de, \tag{7.89}$$

where e is a variable region, and de is an element (differential) of its measure. It follows hence, that the value of an additive and continuously differentiable function $F(e)$ of the variable region e in the given region E equals the integral of the derivative of the given function taken over the region E:

$$F(E) = \int_E \frac{dF(e)}{de}\, de = \int_E dF(e). \tag{7.90}$$

This equation represents an analogue of the well-known Newton–Leibniz formula for integrals.

In actual cases, if we select as the region E in cartesian coordinates:

(1) the line l in the plane Oxy;

(2) the line L in space $Oxyz$;

(3) the region D in the plane Oxy;

(4) the surface S in the space $Oxyz$;

(5) the region V in the space $Oxyz$;

then the corresponding integrals over the measure of the region are denoted and named as follows (see Chapter V):

(1)
$$\int_l f(P)\, ds = \int_l f(x, y)\sqrt{dx^2 + dy^2} \tag{7.91}$$

is the *curvilinear integral over the length of the line l.*

(2)
$$\int_L f(P)\, ds = \int_L f(x, y, z)\sqrt{dx^2 + dy^2 + dz^2} \tag{7.92}$$

is the *curvilinear integral over the length of the line L.*

(3)
$$\iint_D f(P)\, dq = \iint_D f(x, y)\, dx\, dy \tag{7.93}$$

is *the double integral over the area of the region D.*

(4)
$$\iint_S f(P)\, d\sigma$$
$$= \iint_S f(x, y, z)\sqrt{(dx\, dy)^2 + (dy\, dz)^2 + (dz\, dx)^2} \tag{7.94}$$

is *the surface integral over the area of the surface S.*

(5) $$\iiint_V f(P)\,dv = \iiint_V f(x, y, z)\,dx\,dy\,dz \tag{7.95}$$

is *the triple integral over the volume of the region V*.

The integrals over the measure in the cases indicated are calculated as follows:

(1) If the line l is given by the equations

$$x = \varphi(u), \quad y = \psi(u),$$

where φ and ψ are differentiable functions of parameter u in the interval $[u_1, u_2]$, $u_1 < u_2$, then

$$\int_l f(P)\,ds = \int_{u_1}^{u_2} f[\varphi(u), \psi(u)]\sqrt{\varphi'^2(u) + \psi'^2(u)}\,du. \tag{7.96}$$

(2) If the line L is given by the equations

$$x = \varphi(u), \quad y = \psi(u), \quad z = \chi(u),$$

where φ, ψ and χ are differentiable functions of parameter u in the interval $[u_1, u_2]$, $u_1 < u_2$, then

$$\int_L f(P)\,ds$$

$$= \int_{u_1}^{u_2} f[\varphi(u), \psi(u), \chi(u)]\sqrt{\varphi'^2(u) + \psi^2(u) + \chi'^2(u)}\;du. \tag{7.97}$$

(3) If the region D possesses the property, that any straight line parallel to the axis Oy intersects its boundary in not more than two points (at the point of "entry" into the region D and at the point of "exit" out of it), then

$$\iint_D f(P)\,dq = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} f(x, y)\,dy, \tag{7.98}$$

where $y_1(x)$ and $y_2(x)$ are the corresponding ordinates of the points of "entry" and "exit" for the line $x = \text{const}$, $y_1(x) \leqslant y_2(x)$ and the interval $[x_1, x_2]$, $x_1 < x_2$, is the greatest interval of variation of the abscissa x in the region D. The twofold integral, to which the

given integral is reduced, is to be found in the right-hand side. The reduction of a double integral to a twofold one can be carried out also by other methods.

(4) If the surface S is given by the equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

where the right-hand sides contain differentiable functions of parameters u and v in a certain region Δ, then

$$\iint_S f(P)\, d\sigma = \iint_\Delta f[x(u, v), y(u, v), z(u, v)] \sqrt{EG - F^2}\, du\, dv, \qquad (7.99)$$

where E, F, G are Gauss' coefficients of the surface S for the coordinates u and v. Further, the double integral in the right-hand side is calculated in the way described in case (3).

(5) If the region V possesses the property, that any straight line parallel to the axis Oz intersects its boundary in not more than two points (at the point of "entry" into the region V and at the point of "exit" out of it) and the projection of V onto a plane possesses the property indicated in case (3) for the plane region D, then

$$\iiint_V f(P)\, dv = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\, dz, \qquad (7.100)$$

where $z_1(x, y)$ and $z_2(x, y)$ are the corresponding coordinates of the points of "entry" and "exit" of the line $x = \text{const}$, $y = \text{const}$, $z_1(x, y) \leqslant z_2(x, y)$, $y_1(x)$ and $y_2(x)$ are the corresponding ordinates of the points of "entry" and "exit" for the projection of the region V onto the plane Oxy of the line $x = \text{const}$, $z = 0$, and the interval $[x_1, x_2]$, $x_1 < x_2$, is the greatest interval of change of the abscissa x in the region V. The threefold integral, to which the triple integral is reduced, is to be found in the right-hand side. The reduction of the triple integral to a threefold one can also be carried out in other ways.

If in the cases considered the domain of integration does not answer the enumerated requirements, it can be split up into separate

parts, each of which obeys these requirements (assuming that this can be done), after which the additive property of the integral is applied.

Change of Variable in the Integral

2. The integrals (7.91) and (7.92) are reducible to ordinary ("one dimensional" definite) integrals, and (7.94) is reducible to a double integral. Therefore, the question about change of variable can be confined to considering how it is applied to the following three integrals:

$$\int_l f(x)\,dx, \quad \iint_D f(x, y)\,dx\,dy, \quad \iiint_V f(x, y, z)\,dx\,dy\,dz.$$

Suppose the variables (either x, or x, y, or x, y, z) are changed to new ones (either u, or u, v, or u, v, w) according to formulae

$$\left.\begin{aligned} &x = \varphi(u), \\ &x = \varphi(u, v), \quad y = \psi(u, v), \\ &x = \varphi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w). \end{aligned}\right\} \tag{7.101}$$

These formulae for the change of variables can be regarded as formulae for the mapping of the domain of integration $E_1(\lambda, \Delta, \Omega)$ in the cartesian system $Ou\ldots$ into the domain of integration $E(l, D, V)$ in the cartesian system $Ox\ldots$ Here it is assumed that this mapping is homeomorphic and is continuously differentiable, and that if its Jacobian does become zero, then only in a finite number of points.

Let the point Q of the region E_1 be the map of the point P of the region E and the function $f_1(Q)$ is the result of the transformation of function $f(P)$ to new variables. Then, in general,

$$\int_E f(P)\,de = \int_{E_1} f_1(Q)\,K(Q)\,de_1, \tag{7.102}$$

where $K(Q)$ is the coefficient of distortion for the chosen mapping, and de_1 is the element of measure in the region E_1. Therefore, in

particular,

$$\left.\begin{aligned}
&\int_I f(x)\,dx = \int_\lambda f_1(u)\,K(u)\,du,\\
&\iint_D f(x,y)\,dx\,dy = \iint_\Delta f_1(u,v)\,K(u,v)\,du\,dv,\\
&\iiint_V f(x,y,z)\,dx\,dy\,dz\\
&\qquad = \iiint_\Omega f_1(u,v,w)\,K(u,v,w)\,du\,dv\,dw.
\end{aligned}\right\} \tag{7.103}$$

Further, it is known (see Chapter IV) that $KQ = \left|\dfrac{\partial(x, \ldots)}{\partial(u, \ldots)}\right|$, where $\dfrac{\partial(x, \ldots)}{\partial(u, \ldots)}$ is the Jacobian of mapping. Therefore,

$$\left.\begin{aligned}
&\int_I f(x)\,dx = \int_\lambda f_1(u)\left|\frac{dx}{du}\right| du,\\
&\iint_D f(x,y)\,dx\,dy = \iint_\Delta f_1(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,\\
&\iiint_V f(x,y,z)\,dx\,dy\,dz\\
&\qquad = \iiint_\Omega f_1(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.
\end{aligned}\right\} \tag{7.104}$$

If formulae of change of variables define a system of curvilinear orthogonal coordinates, then

(1) $$\int_l f(P)\,ds = \int_l f_1(u,v)\sqrt{l_u^2du^2 + l_v^2dv^2}, \tag{7.105}$$

(2) $$\int_L f(P)\,ds = \int_L f_1(u,v,w)\sqrt{L_u^2du^2 + L_v^2dv^2 + L_w^2dw^2}, \tag{7.106}$$

(3) $$\iint_D f(P)\,dq = \iint_D f_1(u,v)\,l_u l_v\,du\,dv, \tag{7.107}$$

(4) $$\iint_S f(P)\,d\sigma$$

$$= \iint_S f_1(u, v, w)\sqrt{(L_u L_v\,du\,dv)^2 + (L_u L_w\,du\,dw)^2 + (L_v L_w\,dv\,dw)^2}, \tag{7.108}$$

(5) $$\iiint_V f(P)\,dv = \iiint_V f_1(u, v, w)\,L_u L_v L_w\,du\,dv\,dw, \tag{7.109}$$

where l_u, l_v and L_u, L_v, L_w are the corresponding Lamé coefficients, and dv in the integral on the left denotes an element of volume, and du, dv, dw in the integral on the right denote the differentials of the variables.

Integrals with Respect to a Coordinate

3. Integrals with respect to a coordinate are considered mainly in those three cases in which the domain of integration is (1) a line l in a plane, (2) a line L in space, (3) a surface S in space.

Integrals with respect to a coordinate possess the same properties as integrals over the measure. In addition, however, they have the following property: the change of orientation of the region of integration changes the sign of the integral to the opposite one:

(1) $$\int_L f(P)\,dx = -\int_{-L} f(P)\,dx, \tag{7.110}$$

where L and $-L$ denote the same line, but with opposite orientation;

(2) $$\iint_S f(P)\,dx\,dy = -\iint_{-S} f(P)\,dx\,dy, \tag{7.111}$$

where S and $-S$ denote the same surface, but with opposite orientation.

Integrals with respect to a coordinate are calculated as follows:

(1) If the line l is given in terms of parametric equations

$$x = \varphi(u), \quad y = \psi(u), \quad u_1 \leqslant u \leqslant u_2,$$

where φ and ψ are differentiable functions of the variable u, then

$$\int_l f(P)\,dx = \int_{u_1}^{u_2} f[\varphi(u), \psi(u)]\,\varphi'(u)\,du. \tag{7.112}$$

(2) If the line L is given by parametric equations

$$x = \varphi(u), \quad y = \psi(u), \quad z = \chi(u), \quad u_1 \leqslant u \leqslant u_2,$$

where φ, ψ, χ are differentiable functions of the variable u, then

$$\int_L f(P)\,dx = \int_{u_1}^{u_2} f[\varphi(u), \psi(u), \chi(u)]\,\varphi'(u)\,du. \tag{7.113}$$

(3) If the surface S is given by parametric equations

$$x = \varphi(u, v), \quad y = \psi(u, v), \quad z = \chi(u, v),$$

where φ, ψ, χ are differentiable functions of variables u, v and of the oriented region Δ of the plane Ouv, corresponding to the given oriented surface S, then

$$\iint_S f(P)\,dx\,dy = \iint_\Delta f[\varphi(u, v), \psi(u, v), \chi(u, v)]\,\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv, \tag{7.114}$$

where $\partial(x, y)/\partial(u, v)$ is the Jacobian of x, y with respect to u, v.

Curvilinear integrals in other coordinates such as

$$\int_l f(P)\,dy,$$

$$\int_L f(P)\,dy, \quad \int_L f(P)\,dz,$$

$$\iint_S f(P)\,dx\,dz, \quad \iint_S f(P)\,dy\,dz$$

are calculated in a similar way.

Composite curvilinear integrals with respect to coordinates are often applied:

(1) $$\int_l X\,dx + Y\,dy \left(= \int_l X\,dx + \int_l Y\,dy \right), \tag{7.115}$$

(2) $$\int_L X\,dx + Y\,dy + Z\,dz \left(= \int_L X\,dx + \int_L Y\,dy + \int_L Z\,dz \right). \tag{7.116}$$

(3) $$\iint_S X\,dy\,dz + Y\,dx\,dz + Z\,dx\,dy$$

$$\left(= \iint_S X\,dy\,dz + \iint_S Y\,dx\,dz + \iint_S Z\,dx\,dy\right), \quad (7.117)$$

where functions

in (1): $X = X(x, y)$ and $Y = Y(x, y)$,

in (2), (3): $X = X(x, y, z)$, $Y = Y(x, y, z)$ and $Z = Z(x, y, z)$

are continuous in the domain of integration.

§ 3. Formulae for Transformation of Integrals

Green's Fundamental Formula

1. Suppose we are given a finite plane region D bounded by the line l. If the functions $X(x, y)$ and $Y(x, y)$ together with their derivatives of the first order are continuous in the region D, the following *Green's fundamental formula* takes place:

$$\iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\,dy = \int_l X\,dx + Y\,dy, \quad (7.118)$$

where integration with respect to l is carried out in the positive direction. The region D may be multiply connected.

If $(\widehat{n, x})$, $(\widehat{n, y})$ are angles formed by the normal to the line l directed *towards the outside* and the axes Ox and Oy, then, from Green's fundamental formula, we can obtain another Green's formula, of a "symmetric" form:

$$\iint_D \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) dx\,dy$$

$$= \int_l [X \cos(\widehat{n, x}) + Y \cos(\widehat{n, y})\,ds, \quad (7.119)$$

where ds is an element (differential) of the length of the line l.

Green's formula is a convenient aid in calculating certain curvilinear integrals over a closed contour, transforming them into double ones.

EXAMPLE 1. Calculate the integral

$$\int_C -x^2y\,dx + xy^2\,dy,$$

where C is the circumference $x^2 + y^2 = R^2$ traversed in the positive direction.

Here

$$X = -x^2y, \quad Y = xy^2.$$

Correspondingly,

$$\frac{\partial X}{\partial Y} = -x^2, \quad \frac{\partial Y}{\partial x} = y^2$$

and, therefore,

$$\int_C -x^2y\,dx + xy^2\,dy = \int\int_D (y^2 + x^2)\,dx\,dy,$$

where D is the circle $x^2 + y^2 \leqslant R^2$.

The double integral obtained can be calculated by the repeated integration in cartesian coordinates (see p. 167), but in this particular case it is simpler to pass on to polar coordinates. Using the second formula (7.104) and the expression for an element of area in polar coordinates (4.59) we find

$$\int\int_D (x^2 + y^2)\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^R \varrho^3\,d\varrho = \frac{\pi R^4}{2}.$$

Green's formula can be applied only in the case when the functions X and Y together with their partial derivatives $\partial X/\partial y$ and $\partial Y/\partial x$ are continuous in the region D. If these conditions are infringed, the application of Green's formula may lead to incorrect results.

EXAMPLE 2. Calculate the integral

$$\int_C \frac{x\,dy - y\,dx}{x^2 + y^2},$$

where C is any closed curve.

If we apply Green's formula, we get, as a result of the equation

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x},$$

that the integral equals zero. This result holds only when the origin of the coordinates does not lie inside the region D bounded by the curve C. If the region D bounded by the curve C does contain the origin the integral is non-zero. For example, for the circle

$$x = \cos t, \quad y = \sin t, \quad 0 \leqslant t \leqslant 2\pi,$$

we obtain directly

$$\int_C \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_0^{2\pi} dt = 2\pi.$$

Stokes' Formula

2. Suppose the functions $X(x, y, z)$, $Y(x, y, z)$ and $Z(x, y, z)$, together with their partial derivatives of the first order, are continuous in the space region V containing a given surface S with the boundary L. Then *Stokes' formula* holds:

$$\int\int_S \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) dy\, dz + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) dx\, dz + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) dx\, dy = \int_L X\, dx + Y\, dy + Z\, dz. \quad (7.120)$$

Here the orientation of the surface S and the line L are coordinated in such a way, that when a point, taken on the side of the surface, which is undergoing integration, moves along L in the direction of curvilinear integration, the surface S finds itself on the left.

Stokes' formula can be written down by means of integrals over measure, namely

$$\int\int_S \left[\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) \cos\left(\widehat{n, x}\right) + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) \cos\left(\widehat{n, y}\right) + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) \cos\left(\widehat{n, z}\right)\right] dq$$
$$= \int_L \left[X \cos\left(\widehat{s, x}\right) + Y \cos\left(\widehat{s, y}\right) + Z \cos\left(\widehat{s, z}\right)\right] ds, \quad (7.121)$$

where $\left(\widehat{n, x}\right)$, $\left(\widehat{n, y}\right)$, $\left(\widehat{n, z}\right)$ are angles, formed by the oriented normal to the surface S with the axes Ox, Oy, Oz, and $\left(\widehat{s, x}\right)$, $\left(\widehat{s, y}\right)$, $\left(\widehat{s, z}\right)$ are angles, formed by the oriented tangent to the line L with these axes.

In order that the composite curvilinear integral

$$\int_L X\, dx + Y\, dy + Z\, dz$$

be independent of the contour of integration L, which belongs to the simply-connected region V, or, which is equivalent, in order that

this integral equals zero along any closed contour belonging to V, *it is necessary and sufficient* that the functions X, Y, Z, which have continuous partial derivatives of the first order, should satisfy at all points of the region V the relationships:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \quad \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}, \quad \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}. \tag{7.122}$$

The fulfilment of these equations serves as the *necessary and sufficient* condition for the differential expression $Xdx + Ydy + Zdz$ to be a *complete differential* of some function $u = u(x, y, z)$:

$$X\,dx + Y\,dy + Z\,dz = du. \tag{7.123}$$

Here, the function u is called the *original* of the expression $Xdx + Ydy + Zdz$ and can be found from the formula

$$u(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} X\,dx + Y\,dy + Z\,dz, \tag{7.124}$$

where the integral is taken along any path belonging to V and joining points (x_0, y_0, z_0) and (x, y, z).

If the function $F(x, y, z)$ is any one of the originals for $Xdx + Ydy + Zdz$, i.e. if $dF = Xdx + Ydy + Zdz$, then

$$\int_{(x_0, y_0, z_0)}^{(x, y, z)} dF = F(x, y, z) - F(x_0, y_0, z_0).$$

The latter formula serves as an analogue of the Newton–Leibniz formula for composite curvilinear integrals in space.

The Ostrogradskii Formula

3. Suppose, V is a given space region bounded by a closed surface S. If $X(x, y, z)$, $Y(x, y, z)$, $Z(x, y, z)$ are functions, which, together with their partial derivatives of the first order, are continuous in the region V, the following *Ostrogradskii formula* holds:

$$\iiint_V \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dv$$
$$= \iint_S X\,dy\,dz + Y\,dx\,dz + Z\,dx\,dy, \tag{7.125}$$

where the integration over the surface S is carried out on its positive (outer) side.

This formula can be given a different form, namely:

$$\iiint_V \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dx\,dy\,dz$$
$$= \iint_S \left[X \cos(\widehat{n,x}) + Y \cos(\widehat{n,y}) + Z \cos(\widehat{n,z})\right] dq, \qquad (7.126)$$

where $(\widehat{n,x})$, $(\widehat{n,y})$, $(\widehat{n,z})$ are angles formed by the outer normal to the surface S with the axes Ox, Oy, Oz.

For the independence of the composite integral over a surface

$$\iint_S X\,dy\,dz + Y\,dx\,dz + Z\,dx\,dy$$

from the surface S, belonging to the simply-connected region V, or, which is equivalent, for that integral to equal zero over any closed surface belonging to V, *it is necessary and sufficient*, that the functions X, Y, Z, which have continuous partial derivatives of the first order, should, at all points of the region V, satisfy the relationship

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0.$$

Green's Formulae and their Generalisations

4. We quote below a group of formulae called *Green's formulae*.

(a) *The linear case.* If the functions $u(x)$ and $v(x)$ are continuously differentiable twice in the interval $[x_1, x_2]$, the following *Green's formula* holds

$$\int_{x_1}^{x_2} \left(u \frac{d^2 v}{dx^2} - v \frac{d^2 u}{dx^2}\right) dx = \left(u \frac{dv}{dx} - v \frac{du}{dx}\right)\Bigg|_{x_1}^{x_2}. \qquad (7.127)$$

In order to generalize it, we introduce the self-conjugate operator $\mathscr{L}(\varphi)$ of the second order:

$$\mathscr{L}(\varphi) = \frac{d}{dx}\left(A \frac{d\varphi}{dx}\right) + C\varphi, \qquad (7.128)$$

where A, C, φ are some functions given in the interval $[x_1, x_2]$. If we retain the requirements laid down above for the functions u and v, the following formula holds for them then:

$$\int_{x_1}^{x_2} [u\,\mathscr{L}(v) - v\,\mathscr{L}(u)]dx = A\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_{x_1}^{x_2}. \qquad (7.129)$$

This formula is called *Green's generalized formula in the linear case.*

(b) *The plane case.* Suppose $u(x, y)$ and $v(x, y)$ are functions, continuously differentiable twice in the region D, bounded by a closed line l. Then the following *Green's formula* holds:

$$\iint_D (u\,\Delta v - v\,\Delta u)\,dx\,dy = \int_l \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds, \qquad (7.130)$$

where Δu and Δv are Laplacians at points of the region D, and $\partial u/\partial n$, $\partial v/\partial n$ are derivatives of functions u and v in the direction of the outer normal to the line l.

For its generalization the self-conjugate operator $\mathscr{L}(\varphi)$ of the second order is introduced:

$$\mathscr{L}(\varphi) = \frac{\partial}{\partial x}\left(A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y}\right) + F\varphi, \qquad (7.131)$$

where A, B, C, F, φ are some functions given in the region D. If we retain the requirements, laid down above for the functions u and v, the following formula holds for them:

$$\iint_D [u\,\mathscr{L}(v) - v\,\mathscr{L}(u)]\,dx\,dy$$
$$= \int_l \left[X\cos(\widehat{n,x}) + Y\cos(\widehat{n,y})\right] ds, \qquad (7.132)$$

where $(\widehat{n,x})$, $(\widehat{n,y})$ are angles formed by the outer normal to the line l and the axes Ox and Oy respectively, and X and Y are determined from formulae

$$\left.\begin{aligned} X &= A\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) + B\left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right), \\ Y &= B\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) + C\left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right). \end{aligned}\right\} \qquad (7.133)$$

This is *Green's generalized formula in the plane case.*

(c) *Space case*. Suppose $u(x, y, z)$ and $v(x, y, z)$ are functions, continuously differentiable twice in the space region V, bounded by a closed surface S. Then the following *Green's formula* holds:

$$\iiint_V (u\,\Delta v - v\Delta u)\,dx\,dy\,dz = \iint_S \left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right) dq, \tag{7.134}$$

where Δu, Δv are Laplacians at points of the region V, and $\partial u/\partial u$, $\partial v/\partial u$ ***are derivatives of the functions u and v in the direction of the outer normal to the surface S.***

For the generalization of this formula the self-conjugate operator $L(\varphi)$ of the second order is introduced:

$$\mathscr{L}(\varphi) = \frac{\partial}{\partial x}\left(A\,\frac{\partial\varphi}{\partial x} + D\,\frac{\partial\varphi}{\partial y} + E\,\frac{\partial\varphi}{\partial z}\right)$$

$$+ \frac{\partial}{\partial y}\left(D\,\frac{\partial\varphi}{\partial x} + B\,\frac{\partial\varphi}{\partial y} + F\,\frac{\partial\varphi}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z}\left(E\,\frac{\partial\varphi}{\partial x} + F\,\frac{\partial\varphi}{\partial y} + C\,\frac{\partial\varphi}{\partial z}\right) + H\varphi, \tag{7.135}$$

where A, B, C, D, E, F, G, H, φ are some functions given in the region V. If we retain the requirements laid down above for the functions u and v, the following formula holds for them:

$$\iiint_V [u\,\mathscr{L}(v) - v\,\mathscr{L}(u)]\,dx\,dy\,dz$$

$$= \iint_S \left[X\cos\left(\widehat{n, x}\right) + Y\cos\left(\widehat{n, y}\right) + Z\cos\left(\widehat{n, z}\right)\right] dq, \tag{7.136}$$

where $\left(\widehat{u, x}\right)$, $\left(\widehat{u, y}\right)$, $\left(\widehat{u, z}\right)$ are angles, formed by the outer normal to the surface S and the axes Ox, Oy, Oz respectively, and X, Y, Z

are determined from the formulae

$$\left.\begin{aligned}X &= A\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) + D\left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right)\\ &\quad + E\left(u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z}\right),\\ Y &= D\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) + B\left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right)\\ &\quad + F\left(u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z}\right),\\ Z &= E\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) + F\left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right)\\ &\quad + C\left(u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z}\right).\end{aligned}\right\} \quad (7.137)$$

This is *Green's generalized formula in the space case.*

APPENDIXES

As a supplement to the main text we append tables of derivatives (of the first and of the nth orders), of expansions into power series and of integrals (definite, indefinite and multiple) of elementary functions. We also add numerical tables of special functions defined by integrals.

1. Derivatives of Elementary Functions

Derivatives of the first order are given in Table A. The first column of the table contains the set X over which the function $u(x)$, contained in the second column, is differentiated. The third column contains the derivative $u'(x)$, and the fourth shows the simplest relationships between the function $u(x)$ and its derivative $u'(x)$ (differential equations which are satisfied by $u(x)$). If X coincides with the entire numerical straight line E_1, the notation of X is omitted in the first column.

Table B contains general expressions of derivatives of the nth order of certain elementary functions.

TABLE A. DERIVATIVES OF THE FIRST ORDER

X	$u(x)$	$u'(x)$	Differential equation for $u(x)$
$x \geqslant 0$ for non-integral α	x^α	$\alpha x^{\alpha-1}$	$u' = \frac{\alpha u}{x}$
	e^x	e^x	$u' = u$
	a^x	$a^x \ln a$	$u' = u \ln a$
$x > 0$	$\ln x$	$\frac{1}{x}$	$u' = e^{-u}$
$x \neq 0$	$\ln \lvert x \rvert$	$\frac{1}{x}$	$u' = e^{-u}$

TABLE A *(contd.)*

X	$u(x)$	$u'(x)$	Differential equation for $u(x)$
$x > 0$	$\log_a x \quad (a > 0)$	$\dfrac{1}{x \ln a}$	$u' = \dfrac{a^{-u}}{\ln a}$
		$\dfrac{\log_a e}{x}$	$u' = a^{-u} \log_a e$
	$\sin x$	$\cos x$	$u^2 + (u')^2 = 1$
	$\cos x$	$-\sin x$	
$x \neq \dfrac{\pi}{2} + \pi n$ $(n = 0, \pm 1, \pm 2, \ldots)$	$\tan x$	$\dfrac{1}{\cos x^2}$	$u' = 1 + u^2$
$x \neq \pi n$ $(n = 0), \pm 1, \pm 2, \ldots)$	$\cot x$	$-\dfrac{1}{\sin^2 x}$	$u' = -(1 + u^2)$
$\lvert x \rvert < 1$	$\arcsin x$	$\dfrac{1}{\sqrt{1 - x^2}}$	$u' = \dfrac{1}{\cos u}$
$\lvert x \rvert < 1$	$\arccos x$	$-\dfrac{1}{\sqrt{1 - x^2}}$	$u' = -\dfrac{1}{\sin u}$
	$\arctan x$	$\dfrac{1}{1 + x^2}$	$u' = \cos^2 u$
	$\operatorname{arccot} x$	$-\dfrac{1}{1 + x^2}$	$u' = -\sin^2 u$
	$\sinh x$	$\cosh x$	$(u')^2 = u^2 + 1$
	$\cosh x$	$\sinh x$	$(u')^2 = u^2 - 1$
	$\tanh x$	$\dfrac{1}{\cosh^2 x}$	$u' = 1 - u^2$
$x \neq 0$	$\coth x$	$-\dfrac{1}{\sinh^2 x}$	
	$\operatorname{Arcsinh} x = \ln (x + \sqrt{x^2 + 1})$	$\dfrac{1}{\sqrt{x^2 + 1}}$	$u' = \dfrac{1}{\cosh u}$
$\lvert x \rvert > 1$	$\operatorname{Arccosh} x = \ln (x + \sqrt{x^2 - 1})$	$\dfrac{1}{\sqrt{x^2 - 1}}$	$u' = \dfrac{1}{\sinh u}$
$\lvert x \rvert < 1$	$\operatorname{Arctanh} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$	$\dfrac{1}{1 - x^2}$	$u' = \cosh^2 u$
$\lvert x \rvert > 1$	$\operatorname{Arccoth} x = \dfrac{1}{2} \ln \dfrac{x + 1}{x - 1}$	$-\dfrac{1}{x^2 - 1}$	$u' = -\sinh^2 u$

TABLE B. DERIVATIVES OF THE nth ORDER

$f(x)$	$f^{(n)}(x)$
x^m	$m(m-1)(m-2)\ldots(m-n+1)x^{m-n}$ (when m is integral and $n > m$, the derivative equals zero)
$\frac{1}{x^m}$	$(-1)^n m(m+1)(m+2)\ldots(m+n-1)\frac{1}{x^{m+n}}$
$\sqrt[m]{x}$	$(-1)^{n-1}\frac{1}{m^n}(m-1)(2m-1)\ldots[(n-1)m-1]\frac{1}{\sqrt[m]{x^{mn-1}}}$
$(ax+b)^m$	$m(m-1)(m-2)\ldots(m-n+1)a^n(ax+b)^{m-n}$
e^x	e^x
e^{kx}	$k^n e^{kx}$
a^x	$a^x(\ln a)^n$
a^{kx}	$a^{kx}(k\ln a)^n$
$\ln x$	$(-1)^{n-1}(n-1)!\frac{1}{x^n}$
$\log_a x$	$(-1)^{n-1}\frac{(n-1)!}{\ln a}\frac{1}{x^n}$
$\sin x$	$\sin\left(x+\frac{n\pi}{2}\right) = \begin{cases}(-1)^{\frac{n}{2}}\sin x \text{ for even } n\\ (-1)^{\frac{n-1}{2}}\cos x \text{ for odd } n\end{cases}$
$\cos x$	$\cos\left(x+\frac{n\pi}{2}\right) = \begin{cases}(-1)^{\frac{n}{2}}\cos x \text{ for even } n\\ (-1)^{\frac{n+1}{2}}\sin x \text{ for odd } n\end{cases}$
$\sin kx$	$k^n\sin\left(kx+\frac{n\pi}{2}\right)$
$\cos kx$	$k^n\cos\left(kx+\frac{n\pi}{2}\right)$
$\sinh x$	$\begin{cases}\sinh x & \text{for even } n\\ \cosh x & \text{for odd } n\end{cases}$
$\cosh x$	$\begin{cases}\sinh x & \text{for even } n\\ \cosh x & \text{for odd } n\end{cases}$

2. The Expansion of Elementary Functions into Power Series

The expansion of elementary functions into a power series is given in the first column of the Table C. In the second column, the region of convergence of the series to the function, which is being expanded, is shown.

TABLE C. EXPANSION OF FUNCTIONS INTO A POWER SERIES

Expansion into series	The region of convergence of the series
$(a+x)^m = a^m + ma^{m-1}x + \cdots + \frac{m(m-1)\dots(m-n+1)}{n!}a^{m-n}x^n + \cdots$	$-\lvert a\rvert < x < +\lvert a\rvert$
$a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2(\ln a)^2}{2!} + \cdots + \frac{x^n(\ln a)^n}{n!} + \cdots$	$-\infty < x < +\infty$
$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < +\infty$
$\sin x = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < +\infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$	$-\infty < x < +\infty$
$\tan x = x + \frac{x^3}{3!} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\cot x = \frac{1}{x} - \left(\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945} + \frac{x^7}{4725} + \cdots\right)$	$-\pi < x < \pi$ except for $x = 0$
$\arcsin x = x + \frac{1\times x^3}{2\times 3} + \frac{1\times 3x^5}{2\times 4\times 5} + \frac{1\times 3\times 5x^7}{2\times 4\times 6\times 7} + \cdots + \frac{1\times 3\times 5\dots(2n-1)x^{2n+1}}{2\times 4\times 6\dots(2n)(2n+1)} + \cdots$	$-1 < x < 1$

TABLE C *(contd.)*

Expansion into series	The region of convergence of the series
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$	$-1 \leqslant x \leqslant 1$
$\sinh x = x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < +\infty$
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$	$-\infty < x < +\infty$
$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \cdots$	$-\pi < x < \pi$ except for $x = 0$
$\operatorname{Arcsinh} x = x - \frac{x^3}{2 \times 3} + \frac{1 \times 3x^5}{2 \times 4 \times 5} - \frac{1 \times 3 \times 5x^7}{2 \times 4 \times 6 \times 7} + \cdots + (-1)^n \frac{1 \times 3 \times 5 \ldots (2n-1)x^{2n+1}}{2 \times 4 \times 6 \ldots (2n)(2n+1)} + \cdots$	$-1 \leqslant x \leqslant 1$
$\operatorname{Arctanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$	$-1 < x < 1$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$	$-1 < x \leqslant 1$
$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots + \frac{1}{2n+1}\left(\frac{x-1}{x+1}\right)^{2n+1} + \cdots\right]$	$x > 0$
$\ln \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n+1} + \cdots\right)$	$-1 < x < 1$

TABLE C (contd.)

Expansion into series	The region of convergence of the series
$\ln\frac{x+1}{x-1} = 2\left[\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots + \frac{1}{(2n+1)x^{2n+1}}+\cdots\right]$	$x<-1,\ \ x>1$
$\sinh x + \sin x = 2\left(\frac{x}{1}+\frac{x^5}{5!}+\frac{x^9}{9!}+\cdots\right)$	$-\infty<x<+\infty$
$\sinh x - \sin x = 2\left(\frac{x^3}{3!}+\frac{x^7}{7!}+\frac{x^{11}}{11!}+\cdots\right)$	$-\infty<x<+\infty$
$\cosh x + \cos x = 2\left(1+\frac{x^4}{4!}+\frac{x^8}{8!}+\cdots\right)$	$-\infty<x<+\infty$
$\cosh x - \cos x = 2\left(\frac{x^2}{2!}+\frac{x^6}{6!}+\frac{x^{10}}{10!}+\cdots\right)$	$-\infty<x<+\infty$
$\tanh x + \tan x = 2\left(x+\frac{2}{15}x^5+\frac{62}{2835}x^9+\cdots\right)$	$-\infty<x<+\infty$
$\tanh x - \tan x = -2\left(\frac{x^3}{3}+\frac{17}{315}x^7+\frac{1382}{155925}x^{11}+\cdots\right)$	$-\infty<x<+\infty$

3. Integrals of Elementary Functions

Integrals of elementary functions are given in Table D (indefinite integrals), in Table E (definite integrals) and in Table F (multiple integrals). For more detailed tables, see [4], [10] and [22].

TABLE D. INDEFINITE INTEGRALS†

1. $\displaystyle\int (ax+b)^{\nu}\,dx = \frac{1}{(\nu+1)a}(ax+b)^{\nu+1}+C \qquad (\nu \neq -1).$

2. $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|+C.$

† If several expressions are given for one integral, the C may be different.

3. $$\int \frac{P_n(x)}{(x-a)^{n+1}}\,dx$$

$$= -\sum_{k=0}^{n-1} \frac{P_n^{(k)}(a)}{k!(n-k)(x-a)^{n-k}} + \frac{P_n^{(n)}(a)}{n!}\ln|x-a| + C$$

($P_n(x)$ is a polynomial of the nth degree)

4. $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C = -\frac{1}{a}\arctan\frac{x}{a} + C \qquad (a \neq 0).$$

5. $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$= -\frac{1}{a}\operatorname{Arccoth}\frac{x}{a} + C \qquad (|x| > a,\ a \neq 0).$$

6. $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

$$= \frac{1}{a}\operatorname{Arctanh}\frac{x}{a} + C \qquad (|x| < a,\ a \neq 0).$$

7. $$\int \frac{dx}{(a^2 \pm x^2)^n} = \frac{x}{2(n-1)a^2(a^2 \pm x^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(a^2 \pm x^2)^{n-1}}$$

$$(n \neq 1, \quad a \neq 0).$$

8. $$\int \frac{dx}{ax^2+b} = \frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}\,x + C \qquad (ab > 0).$$

9. $$\int \frac{dx}{ax^2-b} = \frac{1}{2\sqrt{ab}}\ln\left|\frac{\sqrt{ab}-ax}{\sqrt{ab}+ax}\right| + C \qquad (ab > 0).$$

10. $$\int \frac{dx}{ax^2+bx+c}$$

$$= \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}} + C & (4ac-b^2 > 0), \\[2ex] \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (4ac-b^2 < 0), \\[2ex] -\dfrac{2}{2ax+b} + C & (4ac-b^2 = 0). \end{cases}$$

11. $$\int \frac{dx}{(ax^2+bx+c)^n} = \frac{2ax+b}{(n-1)(4ac-b^2)(ax^2+bx+c)^{n-1}}$$

$$+ \frac{2(2n-3)a}{(n-1)(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^{n-1}} \qquad (n \neq 1,\ a \neq 0).$$

12. $$\int \frac{Mx+N}{(x+a)(x+b)}\,dx$$
$$= \frac{1}{b-a}[(N-Ma)\ln|x-a| - (N-Mb)\ln|x+b|] + C \qquad (a \neq b).$$

13. $$\int \frac{Mx+N}{ax^2+bx+c}\,dx$$
$$= \frac{M}{2a}\ln|ax^2+bx+c| + \frac{2aN-Mb}{2a}\int \frac{dx}{ax^2+bx+c} \qquad (a \neq 0).$$

14. $$\int \frac{Mx+N}{(ax^2+bx+c)^n}\,dx = -\frac{M}{2(n-1)\,a(ax^2+bx+c)^{n-1}}$$
$$+ \frac{2aN-bM}{2a}\int \frac{dx}{(ax^2+bx+c)^n} \qquad (n \neq 1,\quad a \neq 0).$$

15. $$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C.$$

16. $$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + C$$
$$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\operatorname{Arcsinh}\frac{x}{a} + C.$$

17. $$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$$
$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\operatorname{Arccosh}\frac{x}{a} + C.$$

18. $$\int \sqrt{ax^2+bx+c}\,dx$$
$$= \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

19. $$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C.$$

20. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}| + C = \operatorname{Arcsinh}\frac{x}{a} + C.$$

21. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + C = \operatorname{Arccosh}\frac{x}{a} + C.$$

22. $$\int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}} = -\frac{n}{a-b}\sqrt[n]{\frac{x-b}{x-a}} + C$$

(n is a natural number, $a \neq b$).

23. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \ln |2ax + b + 2\sqrt{a}\,\sqrt{ax^2+bx+c}| + C \qquad (a > 0),$$

$$= \frac{1}{\sqrt{a}} \operatorname{Arcsinh} \frac{2ax+b}{\sqrt{4ac-b^2}} + C \qquad (a > 0,\quad 4ac - b^2 > 0),$$

$$= -\frac{1}{\sqrt{-a}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} + C \qquad (a < 0,\quad 4ac - b^2 < 0),$$

$$= \frac{1}{\sqrt{a}} \ln |2ax + b| + C \qquad (a > 0,\quad 4ac - b^2 = 0).$$

24. $$\int \frac{Mx+N}{\sqrt{ax^2+bx+c}}\,dx = \frac{M}{a}\sqrt{ax^2+bx+c} + \frac{2aN - Mb}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

25. $$\int \frac{x^n\,dx}{\sqrt{ax^2+bx+c}} = \frac{x^{n-1}\sqrt{ax^2+bx+c}}{na} - \frac{(n\;1)c}{na}\int \frac{x^{n-2}\,dx}{\sqrt{ax^2+bx+c}} - \frac{(2n-1)b}{2na}\int \frac{x^{n-1}\,dx}{\sqrt{ax^2+bx+c}}.$$

26. $$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C.$$

27. $$\int a^{kx}\,dx = \frac{a^{kx}}{k \ln a} + C.$$

28. $$\int \ln x\,dx = x \ln x - x + C.$$

29. $$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C.$$

30. $$\int \cos ax\,dx = \frac{1}{a}\sin ax + C.$$

31. $$\int \tan ax\,dx = -\frac{1}{a}\ln |\cos ax| + C.$$

32. $\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C.$

33. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C.$

34. $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C.$

35. $\int \tanh ax\,dx = \frac{1}{a}\ln|\cosh ax| + C.$

36. $\int \coth ax\,dx = \frac{1}{a}\ln|\sinh ax| + C.$

37. $\int \arcsin\frac{x}{a}\,dx = x\arcsin\frac{x}{a} + \sqrt{a^2 - x^2} + C.$

38. $\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2 - x^2} + C.$

39. $\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2 + x^2) + C.$

40. $\int \operatorname{arccot}\frac{x}{a}\,dx = x\operatorname{arccot}\frac{x}{a} + \frac{a}{2}\ln(a^2 + x^2) + C.$

41. $\int \operatorname{Arcsinh}\frac{x}{a}\,dx = x\operatorname{Arcsinh}\frac{x}{a} - \sqrt{a^2 + x^2} + C.$

42. $\int \operatorname{Arccosh}\frac{x}{a}\,dx = x\operatorname{Arccosh}\frac{x}{a} - \sqrt{x^2 - a^2} + C.$

43. $\int \operatorname{Arctanh}\frac{x}{a}\,dx = x\operatorname{Arctanh}\frac{x}{a} + \frac{a}{2}\ln|a^2 - x^2| + C.$

44. $\int \operatorname{Arccoth}\frac{x}{a}\,dx = x\operatorname{Arccoth}\frac{x}{a} + \frac{a}{2}\ln|x^2 - a^2| + C.$

45. $\int \sinh ax\sin ax\,dx = \frac{1}{2a}(\cosh ax\sin ax - \sinh ax\cos ax) + C.$

46. $\int \cosh ax\cos ax\,dx = \frac{1}{2a}(\sinh ax\cos ax + \cosh ax\sin ax) + C.$

47. $\int \sinh ax\cos ax\,dx = \frac{1}{2a}(\cosh ax\cos ax + \sinh ax\sin ax) + C.$

48. $\int \cosh ax\sin ax\,dx = \frac{1}{2a}(\sinh ax\sin ax - \cosh ax\cos ax) + C.$

49. $$\int \sin ax \sin bx \, dx = \frac{1}{2(a-b)} \sin (a-b)x - \frac{1}{2(a+b)} \sin (a+b)x + C \qquad (a^2 \neq b^2).$$

50. $$\int \cos ax \cos bx \, dx = \frac{1}{2(a-b)} \sin (a-b)x + \frac{1}{2(a+b)} \sin (a+b)x + C \qquad (a^2 \neq b^2).$$

51. $$\int \sin ax \cos bx \, dx = -\frac{1}{2(a+b)} \cos (a+b)x - \frac{1}{2(a-b)} \cos (a-b)x + C \qquad (a^2 \neq b^2).$$

52. $$\int \sinh ax \sinh bx \, dx = \frac{1}{2(a+b)} \sinh (a+b)x - \frac{1}{2(a-b)} \sinh (a-b)x + C \qquad (a^2 \neq b^2).$$

53. $$\int \cosh ax \cosh bx \, dx = \frac{1}{2(a+b)} \sinh (a+b)x + \frac{1}{2(a-b)} \sinh (a-b)x + C \qquad (a^2 \neq b^2).$$

54. $$\int \sinh ax \cosh bx \, dx = \frac{1}{2(a+b)} \cosh (a+b)x + \frac{1}{2(a-b)} \cosh (a-b)x + C \qquad (a^2 \neq b^2).$$

55. $$\int \sinh ax \sin bx \, dx = \frac{1}{a^2+b^2} (a \cosh ax \sin bx - b \sinh ax \cos bx) + C.$$

56. $$\int \cosh ax \cos bx \, dx = \frac{1}{a^2+b^2} (a \sinh ax \cos bx + b \cosh ax \sin bx) + C.$$

57. $$\int \sinh ax \cos bx \, dx = \frac{1}{a^2+b^2} (a \cosh ax \cos bx + b \sinh ax \sin bx) + C.$$

58. $$\int \cosh ax \sin bx \, dx = \frac{1}{a^2+b^2} (a \sinh ax \sin bx - b \cosh ax \cos bx) + C.$$

59. $$\int \left(\arcsin \frac{x}{a}\right)^2 dx = x \left(\arcsin \frac{x}{a}\right)^2 + 2\sqrt{a^2 - x^2} \arcsin \frac{x}{a} - 2x + C.$$

60. $\int \left(\arccos \frac{x}{a}\right)^2 dx = x\left(\arccos \frac{x}{a}\right)^2 - 2\sqrt{a^2 - x^2}\arccos \frac{x}{a} - 2x + C.$

61. $\int (\ln x)^n \, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$

$$= x[(\ln x)^n - n(\ln x)^{n-1} + n(n-1)(\ln x)^{n-2} - \cdots$$

$$+ (-1)^{n-1}n(n-1)\ldots 2\ln x + (-1)^n n!] + C$$

(n is a natural number).

62. $\int \sin^n ax\, dx = -\frac{1}{na}\sin^{n-1} ax \cos ax + \frac{n-1}{n}\int \sin^{n-2} ax\, dx \quad (n \neq 0).$

63. $\int \cos^n ax\, dx = \frac{1}{na}\cos^{n-1} ax \sin ax + \frac{n-1}{n}\int \cos^{n-2} ax\, dx \quad (n \neq 0).$

64. $\int \tan^n ax\, dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\, dx \quad (n \neq 1).$

65. $\int \cot^n ax\, dx = -\frac{\cot^{n-1} ax}{(n-1)a} - \int \cot^{n-2} ax\, dx \quad (n \neq 1).$

66. $\int \sinh^n ax\, dx = \frac{1}{an}\sinh^{n-1} ax \cosh ax$

$$- \frac{n-1}{n}\int \sinh^{n-2} ax\, dx \quad (n \neq 0).$$

67. $\int \cosh^n ax\, dx = \frac{1}{an}\cosh^{n-1} ax \sinh ax$

$$+ \frac{n-1}{n}\int \cosh^{n-2} ax\, dx \quad (n \neq 0).$$

68. $\int \tanh^n ax\, dx = -\frac{\tanh^{n-1} ax}{(n-1)a} + \int \tanh^{n-2} ax\, dx \quad (n \neq 1).$

69. $\int \coth^n ax\, dx = -\frac{\coth^{n-1} ax}{(n-1)a} + \int \coth^{n-2} ax\, dx \quad (n \neq 1).$

70. $\int \frac{dx}{\sin ax} = \frac{1}{a}\ln\left|\tan \frac{ax}{2}\right| + C.$

71. $\int \frac{dx}{\cos ax} = \frac{1}{a}\ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right| + C.$

72. $\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\left|\tanh \frac{ax}{2}\right| + C.$

73. $\int \frac{dx}{\cosh ax} = \frac{2}{a}\arctan e^{ax} + C.$

74. $$\int \frac{dx}{b + c\sin ax} = \frac{2}{a\sqrt{b^2 - c^2}} \arctan \frac{b\tan\frac{ax}{2} + c}{\sqrt{b^2 - c^2}} + C \qquad (b^2 > c^2),$$

$$= \frac{1}{a\sqrt{c^2 - b^2}} \ln \left| \frac{b\tan\frac{ax}{2} + c - \sqrt{c^2 - b^2}}{b\tan\frac{ax}{2} + c + \sqrt{c^2 - b^2}} \right| + C \qquad (b^2 < c^2).$$

75. $$\int \frac{dx}{b + c\cos ax} = \frac{2}{a\sqrt{b^2 - c^2}} \arctan \left(\sqrt{\frac{b - c}{b + c}} \tan\frac{ax}{2} \right) + C \quad (b^2 > c^2),$$

$$= \frac{1}{a\sqrt{c^2 - b^2}} \ln \left| \frac{(c - b)\tan\frac{ax}{2} + \sqrt{c^2 - b^2}}{(c - b)\tan\frac{ax}{2} - \sqrt{c^2 - b^2}} \right| + \quad C \quad (b^2 < c^2).$$

76. $$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax + C.$$

77. $$\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax + C.$$

78. $$\int \frac{dx}{\sinh^2 ax} = -\frac{1}{a}\coth ax + C.$$ 78a. $$\int \frac{dx}{\cosh^2 ax} = \frac{1}{a}\tanh ax + C.$$

79. $$\int \frac{dx}{\sin(x + a)\sin(x + b)} = \frac{1}{\sin(a - b)} \ln \left| \frac{\sin(x + b)}{\sin(x + a)} \right| + C$$

$$[\sin(a - b) \neq 0].$$

80. $$\int \frac{dx}{\cos(x + a)\cos(x + b)} = \frac{1}{\sin(a - b)} \ln \left| \frac{\cos(x + b)}{\cos(x + a)} \right| + C$$

$$[\sin(a - b) \neq 0].$$

81. $$\int \frac{dx}{\sin(x + a)\cos(x + b)} = \frac{1}{\cos(a - b)} \ln \left| \frac{\sin(x + a)}{\cos(x + b)} \right| + C$$

$$[\cos(a - b) \neq 0].$$

82. $$\int \frac{dx}{\sin^n ax} = -\frac{1}{(n - 1)a} \frac{\cos ax}{\sin^{n-1} ax} + \frac{n - 2}{n - 1} \int \frac{dx}{\sin^{n-2} ax} \qquad (n \neq 1).$$

83. $$\int \frac{dx}{\cos^n ax} = \frac{1}{(n - 1)a} \frac{\sin ax}{\cos^{n-1} ax} + \frac{n - 2}{n - 1} \int \frac{dx}{\cos^{n-2} ax} \qquad (n \neq 1).$$

84. $$\int \frac{dx}{\sinh^n ax} = \frac{1}{(1 - n)a} \frac{\cosh ax}{\sinh^{n-1} ax} - \frac{n - 2}{n - 1} \int \frac{dx}{\sinh^{n-2} ax} \qquad (n \neq 1).$$

85. $\displaystyle\int \frac{dx}{\cosh^n ax} = \frac{1}{(n-1)a} \frac{\sinh ax}{\cosh^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} ax} \qquad (n \neq 1).$

86. $$\int \sin^p x \cos^q x \, dx = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x \, dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx \qquad (p \text{ and } q > 0).$$

87. $$\int \sin^{-p} x \cos^q x \, dx = -\frac{\sin^{-p+1} x \cos^{q+1} x}{p-1} + \frac{p-q-2}{p-1} \int \sin^{-p+2} x \cos^q x \, dx \qquad (p \neq 1).$$

88. $$\int \sin^p x \cos^{-q} x \, dx = \frac{\sin^{p+1} x \cos^{-q+1} x}{q-1} + \frac{q-p-2}{q-1} \int \sin^p x \cos^{-q+2} x \, dx \qquad (q \neq 1).$$

89. $\displaystyle\int xe^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1) + C.$

90. $\displaystyle\int x \ln x \, dx = \frac{x^2}{4} (2 \ln x - 1) + C.$

91. $\displaystyle\int x \sin ax \, dx = -\frac{1}{a} x \cos ax + \frac{1}{a^2} \sin ax + C.$

92. $\displaystyle\int x \cos ax \, dx = \frac{1}{a} x \sin ax + \frac{1}{a^2} \cos ax + C.$

93. $\displaystyle\int x \sinh ax \, dx = \frac{1}{a} x \cosh ax - \frac{1}{a^2} \sinh ax + C.$

94. $\displaystyle\int x \cosh ax \, dx = \frac{1}{a} x \sinh ax - \frac{1}{a^2} \cosh ax + C.$

95. $\displaystyle\int x \arcsin \frac{x}{a} \, dx = \frac{1}{4} \left[(2x^2 - a^2) \arcsin \frac{x}{a} + x\sqrt{a^2 - x^2}\right] + C.$

96. $\displaystyle\int x \arccos \frac{x}{a} \, dx = \frac{1}{4} \left[(2x^2 - a^2) \arccos \frac{x}{a} - x\sqrt{a^2 - x^2}\right] + C.$

97. $\displaystyle\int x \arctan \frac{x}{a} \, dx = \frac{1}{2} \left[(x^2 + a^2) \arctan \frac{x}{a} - ax\right] + C.$

98. $\int x \operatorname{arccot} \frac{x}{a}\, dx = \frac{1}{2}\left[(x^2 + a^2) \operatorname{arccot} \frac{x}{a} + ax\right] + C.$

99. $\int x \operatorname{Arcsinh} \frac{x}{a}\, dx = \frac{1}{4}\left[(2x^2 + a^2) \operatorname{Arcsinh} \frac{x}{a} - x\sqrt{a^2 + x^2}\right] + C.$

100. $\int x \operatorname{Arccosh} \frac{x}{a}\, dx$

$$= \begin{cases} \frac{1}{4}\left[(2x^2 - a^2) \operatorname{Arccosh} \frac{x}{a} - x\sqrt{x^2 - a^2}\right] + C & \left(\operatorname{Arccosh} \frac{x}{a} > 0\right), \\ \frac{1}{4}\left[(2x^2 - a^2) \operatorname{Arccosh} \frac{x}{a} + x\sqrt{x^2 - a^2}\right] + C & \left(\operatorname{Arccosh} \frac{x}{a} < 0\right). \end{cases}$$

101. $\int e^{ax} \sin(\omega x + \varphi)\, dx$

$$= \frac{e^{ax}}{a^2 + \omega^2}\left[a \sin(\omega x + \varphi) - \omega \cos(\omega x + \varphi)\right] + C.$$

102. $\int e^{ax} \cos(\omega x + \varphi)\, dx$

$$= \frac{e^{ax}}{a^2 + \omega^2}\left[a \cos(\omega x + \varphi) + \omega \sin(\omega x + \varphi)\right] + C.$$

103. $\int x e^{ax} \sin(\omega x + \varphi)\, dx$

$$= \frac{x e^{ax}}{a^2 + \omega^2}\left[a \sin(\omega x + \varphi) - \omega \cos(\omega x + \varphi)\right]$$

$$- \frac{e^{ax}}{(a^2 + \omega^2)^2}\left[(a - \omega^2) \sin(\omega x + \varphi) - 2a\omega \cos(\omega x + \varphi)\right] + C.$$

104. $\int x e^{ax} \cos(\omega x + \varphi)\, dx$

$$= \frac{x e^{ax}}{a^2 + \omega^2}\left[a \cos(\omega x + \varphi) + \omega \sin(\omega x + \varphi)\right]$$

$$- \frac{e^{ax}}{(a^2 + \omega^2)^2}\left[(a^2 - \omega^2) \cos(\omega x + \varphi) + 2a\omega \sin(\omega x + \varphi)\right] + C.$$

105. $\int x^n e^{ax}\, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx.$

106. $\int x^n \ln x\, dx = x^{n+1}\left[\frac{\ln x}{n + 1} - \frac{1}{(n + 1)^2}\right] + C.$

107. $\int x^n \sin ax\, dx = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\, dx.$

108. $\displaystyle\int x^n \cos ax\,dx = \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1}\sin ax\,dx.$

109. $\displaystyle\int x^n \sinh ax\,dx = \frac{1}{a}x^n \cosh ax - \frac{n}{a}\int x^{n-1}\cosh ax\,dx.$

110. $\displaystyle\int x^n \cosh ax\,dx = \frac{1}{a}x^n \sinh ax - \frac{n}{a}\int x^{n-1}\sinh ax\,dx.$

111. $\displaystyle\int x^n \arcsin\frac{x}{a}\,dx$

$$= \frac{1}{n+1}\left(x^{n+1}\arcsin\frac{x}{a} - \int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}\right) \qquad (n \neq -1).$$

112. $\displaystyle\int x^n \arccos\frac{x}{a}\,dx$

$$= \frac{1}{n+1}\left(x^{n+1}\arccos\frac{x}{a} + \int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}\right) \qquad (n \neq -1).$$

113. $\displaystyle\int x^n \arctan\frac{x}{a}\,dx = \frac{1}{n+1}\left(x^{n+1}\arctan\frac{x}{a} - a\int\frac{x^{n+1}\,dx}{a^2+x^2}\right)$

$(n \neq -1).$

114. $\displaystyle\int x^n \operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{n+1}\left(x^{n+1}\operatorname{arccot}\frac{x}{a} + a\int\frac{x^{n+1}\,dx}{a^2+x^2}\right)$

$(n \neq -1).$

115. $\displaystyle\int P(x)e^{ax}\,dx = e^{ax}\left[\frac{P(x)}{a} - \frac{P'(x)}{a^2} + \cdots + (-1)^n\frac{P^{(n)}(x)}{a^{n+1}}\right] + C$

[$P(x)$—polynomial degree n].

116. $\displaystyle\int P(x)\sin ax\,dx = -\frac{\cos ax}{a}\left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{\mathrm{IV}}(x)}{a^4} - \cdots\right]$

$$+ \frac{\sin ax}{a^2}\left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{\mathrm{V}}(x)}{a^4} - \cdots\right] + C.$$

117. $\displaystyle\int P(x)\cos ax\,dx = \frac{\sin ax}{a}\left[P(x) - \frac{P''(x)}{a^2} + \frac{P^{\mathrm{IV}}(x)}{a^4} - \cdots\right]$

$$+ \frac{\cos ax}{a^2}\left[P'(x) - \frac{P'''(x)}{a^2} + \frac{P^{\mathrm{V}}(x)}{a^4} - \cdots\right] + C.$$

TABLE E. DEFINITE INTEGRALS

118. $$\int_0^\infty \frac{x^m\,dx}{(a+bx^n)^p} = \frac{a^{-p}}{n}\left(\frac{a}{b}\right)^{\frac{m+1}{n}} B\left(\frac{m+1}{n}, p-\frac{m+1}{n}\right)^\dagger$$
$$\left(0<\frac{m+1}{n}<p, \qquad a>0,\ b>0,\ n>0\right).$$

119. $$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{1}{a^n(1+a)^m}B(m,n) \qquad (m>0,\ n>0).$$

120. $$\int_a^b \frac{(x-a)^m(b-x)^n}{(x+c)^{m+n+2}}\,dx$$
$$=\frac{(b-a)^{m+n+1}}{(a+c)^{n+1}(b+c)^{m+1}}B(m+1,\ n+1) \quad (m>-1,\ n>-1).$$

121. $$\int_0^\infty \frac{x^{p-1}-x^{-p}}{1-x}\,dx = \pi\cot\pi p \qquad (0<p<1).$$

122. $$\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n\sin\frac{m\pi}{n}} \qquad (0<m<n).$$

123. $$\int_0^\infty \frac{x^{m-1}}{(1+x)^n}\,dx = B(n-m,m) \qquad (0<m<n).$$

124. $$\int_0^1 \frac{dx}{\sqrt[n]{1-x^m}} = \frac{1}{m}B\left(\frac{1}{m},1-\frac{1}{n}\right) \qquad (n<0 \text{ or } n>1,\ m>0).$$

125. $$\int_0^1 \frac{x\,dx}{\sqrt{1-x^3}} = \frac{\sqrt{3}}{\pi\sqrt[3]{4}}\Gamma^3\left(\frac{2}{3}\right)^{\ddagger}.$$

126. $$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}}\Gamma^3\left(\frac{1}{3}\right).$$

127. $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right).$$

128. $$\int_0^\infty e^{-x^n}\,dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) \qquad (n>0).$$

129. $$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

† $B(\alpha,\beta)$ is Euler's beta-function (see p. 303).
‡ $\Gamma(\alpha)$ is Euler's gamma-function (see p. 303).

130. $\displaystyle\int_0^\infty x^m e^{-x^n}\,dx = \frac{1}{|n|}\Gamma\left(\frac{m+1}{n}\right) \qquad \left(\frac{m+1}{n} > 0\right).$

131. $\displaystyle\int_0^\infty x^n e^{-x}\,dx = n! \qquad (n > 0 \text{ and is an integer}).$

132. $$\int_0^\infty x^n e^{-x^2}\,dx = \begin{cases} \dfrac{(2k-1)!!\sqrt{\pi}}{2^{k+1}} & \text{for} \quad n = 2k, \\[2ex] \dfrac{k!}{2} & \text{for} \quad n = 2k+1. \end{cases}$$

133. $\displaystyle\int_0^\infty \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x}\,dx = \frac{1}{2}\ln\frac{\beta}{\alpha} \qquad (\alpha > 0, \quad \beta > 0).$

134. $\displaystyle\int_0^\infty \left(\frac{e^{-\alpha x} - e^{-\beta x}}{x}\right)^2 dx = \ln\frac{(2\alpha)^{2\alpha}(2\beta)^{2\beta}}{(\alpha+\beta)^{2\alpha+2\beta}} \qquad (\alpha > 0, \quad \beta > 0).$

135. $\displaystyle\int_{-\infty}^\infty e^{-ax^2}\cosh bx\,dx = \sqrt{\frac{\pi}{a}}\cdot e^{b^2/4a} \qquad (a > 0).$

136. $\displaystyle\int_{-\infty}^\infty e^{-ax^2}\sinh^2 bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(e^{b^2/a} - 1\right) \qquad (a > 0).$

137. $\displaystyle\int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{2}e^{-2a} \qquad (a > 0).$

138. $\displaystyle\int_0^\infty \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2}\,dx = \sqrt{\pi}\left(\sqrt{\beta} - \sqrt{\alpha}\right) \qquad (\alpha > 0, \quad \beta > 0).$

139. $\displaystyle\int_0^\infty \frac{\sinh \alpha x}{\sinh \beta x}\,dx = \frac{\pi}{2\beta}\tan\frac{\alpha\pi}{2\beta} \qquad (0 < \alpha < \beta).$

140. $\displaystyle\int_0^1 \left(\ln\frac{1}{x}\right)^p dx = \Gamma(p+1) \qquad (p > -1).$

141. $\displaystyle\int_0^1 \ln(px+q)\,dx = \frac{p+q}{p}\ln(p+q) - \frac{q}{p}\ln q - 1.$

142. $\displaystyle\int_0^1 \ln x \ln(1-x)\,dx = 2 - \frac{\pi^2}{6}.$

143. $\displaystyle\int_0^1 \ln x \ln(1+x)\,dx = 2 - \frac{\pi^2}{12} - 2\ln 2.$

144. $\displaystyle\int_0^1 x^{n-1}\ln(1-x)\,dx = -\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}.$

145. $\int_0^\infty x^p e^{-ax} \ln x \, dx = \frac{d}{dp}\left[\frac{\Gamma(p+1)}{a^{p+1}}\right] \qquad (p > -1,\ a > 0).$

146. $\int_0^1 x^m (\ln x)^n \, dx = (-1)^n \frac{n!}{(m+1)^{n+1}}.$

147. $\int_0^1 \frac{\ln x \, dx}{1-x} = -\frac{\pi^2}{6}.$

148. $\int_0^1 \frac{\ln x \, dx}{1+x} = -\frac{\pi^2}{12}.$

149. $\int_0^1 \frac{\ln x \, dx}{1-x^2} = -\frac{\pi^2}{8}.$

150. $\int_0^1 \frac{\ln(1+x)\, dx}{1+x^2} = \frac{\pi}{8} \ln 2.$

151. $\int_0^\infty \frac{x^{p-1} \ln x}{1+x} \, dx = -\frac{\pi^2 \cos p\pi}{\sin^2 p\pi} \qquad (0 < p < 1).$

152. $\int_0^\infty \frac{x^{p-1} \ln^2 x}{1+x} \, dx = \pi^3 \frac{1+\cos^2 p\pi}{\sin^3 p\pi} \qquad (0 < p < 1).$

153. $\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1+x)\ln x} \, dx = \ln \left| \frac{\tan \frac{p\pi}{2}}{\tan \frac{q\pi}{2}} \right| \qquad (0 < p < 1,\quad 0 < q < 1)$

154. $\int_0^\infty \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \, dx = \frac{\pi}{|\beta|} \ln(|\alpha| + |\beta|) \qquad (\beta \neq 0).$

155. $\int_0^1 \frac{\ln(1-\alpha^2 x^2)}{x^2\sqrt{1-x^2}} \, dx = -\pi\left(1 - \sqrt{1-\alpha^2}\right) \qquad (|\alpha| \leqslant 1).$

156. $\int_0^1 \frac{\ln(1-\alpha^2 x^2)}{\sqrt{1-x^2}} \, dx = \pi \ln \frac{1+\sqrt{1-\alpha^2}}{2} \qquad (|\alpha| \leqslant 1).$

157. $\int_0^\infty \frac{\ln(1+\alpha^2 x^2)\ln(1+\beta^2 x^2)}{x^4} \, dx$

$$= \frac{2\pi}{3}[\alpha\beta(\alpha-\beta) + \alpha^3 \ln \alpha + \beta^3 \ln \beta - (\alpha^3+\beta^3)\ln(\alpha+\beta)]$$

$(\alpha > 0,\quad \beta > 0).$

158. $\int_0^{\pi/2} \sin^{2n+1} x \, dx = \int_0^{\pi/2} \cos^{2n+1} x \, dx$

$$= \frac{2 \times 4 \times 6 \ldots 2n}{3 \times 5 \times 7 \ldots (2n+1)} \qquad (n > 0 \text{ and is an integer}).$$

159. $$\int_0^{\pi/2} \sin^{2n} x\,dx = \int_0^{\pi/2} \cos^{2n} x\,dx$$
$$= \frac{1\times 3\times 5\ldots(2n-1)}{2\times 4\times 6\ldots 2n}\,\frac{\pi}{2} \qquad (n>0 \quad \text{and is an integer}).$$

160. $$\int_0^{\pi/2} \sin^n x\,dx = \frac{1}{2}\,B\left(\frac{n+1}{2},\ \frac{1}{2}\right) \qquad (n>-1).$$

161. $$\int_0^{\pi/2} \sin^{2n+1} x \cos^{2m+1} x\,dx = \frac{m!\,n!}{2(m+n+1)!}$$
$$(m>n,\ n>0,\ m \text{ and } n \text{ are integers}).$$

162. $$\int_0^{\pi/2} \sin^{2\alpha+1} x \cos^{2\beta+1} x\,dx = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2)}.$$

163. $$\int_0^{\pi/2} \sin^{2m} x \cos^{2n} x\,dx = \frac{\pi(2m)!\,(2n)!}{2^{2m+2n+1} m!\,n!\,(m+n)!}$$
$$(m>0,\quad n>0,\quad m \text{ and } n \text{ are integers}).$$

164. $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{m+1}{2},\ \frac{n+1}{2}\right) \qquad (m>-1,\quad n>-1).$$

165. $$\int_0^{\pi/2} \tan^n x\,dx = \frac{\pi}{2\cos\dfrac{n\pi}{2}} \qquad (-1<n<1).$$

166. $$\int_0^{\pi/4} \tan^{2n} x\,dx$$
$$= (-1)^n\left[\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right)\right].$$

167. $$\int_0^{\pi} \sin mx \sin nx\,dx$$
$$= \int_0^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & \text{for } m \neq n \qquad (m, n = 0, \pm 1, \ldots), \\ \dfrac{\pi}{2} & \text{for } m = n. \end{cases}$$

168. $$\int_0^{\pi} \sin mx \cos nx\,dx = 0 \qquad (m \text{ and } n \text{ are integers}).$$

169. $$\int_0^{\pi} \frac{\sin nx}{\sin x}\,dx = \begin{cases} 0 & \text{for } n \text{ even}; \\ \pi & \text{for } n \text{ odd}; \end{cases}$$

170. $$\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x}\,dx = (-1)^n\,\pi.$$

171. $\int_0^{\pi} \sin^n x \sin nx\, dx = \frac{\pi}{2^n} \sin \frac{n\pi}{2}$.

172. $\int_0^{\pi} \cos^n x \cos nx\, dx = \frac{\pi}{2^n}$.

173. $\int_0^{\pi} \sin^{n-1} x \cos (n+1) x\, dx = \int_0^{\pi} \cos^{n-1} x \sin (n+1) x\, dx = 0$.

174. $\int_0^{\pi/2} \sin^{3/2} x\, dx = \int_0^{\pi/2} \cos^{3/2} x\, dx = \frac{1}{6\sqrt{2\pi}} \Gamma^2\left(\frac{1}{4}\right)$.

175. $\int_0^{\pi/2} \sin^{1/3} dx = \frac{3\sqrt{3}}{2\pi \sqrt[3]{4}} \Gamma^3\left(\frac{2}{3}\right)$.

176. $\int_0^{\pi/2} \sin^{-1/3} dx = \frac{\sqrt{3}}{4\pi \sqrt[3]{2}} \Gamma^3\left(\frac{1}{3}\right)$.

177. $\int_0^{\pi/2} \cos^{-1/2} dx = \frac{1}{2\sqrt{2\pi}} \Gamma^2\left(\frac{1}{4}\right)$.

178. $\int_0^{\infty} \sin (x^2)\, dx = \int_0^{\infty} \cos (x^2)\, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$.

179. $\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\, dx = \int_0^{\infty} \frac{\cos x}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{2}}$.

180. $\int_0^{\infty} \frac{\sin \alpha x}{x}\, dx = \frac{\pi}{2} \operatorname{sign} \alpha$ (Dirichlet integral).

181. $\int_0^{\infty} \frac{\cos \alpha x}{x}\, dx = +\infty$.

182. $\int_0^{\infty} \frac{\cos \alpha x - \cos x\beta}{x}\, dx = \ln \frac{\beta}{\alpha} \qquad (\alpha > 0,\ \beta > 0)$.

183. $\int_0^{\infty} \frac{\sin \alpha x - \sin x\beta}{x}\, dx = 0 \qquad (\alpha > 0,\ \beta > 0)$.

184. $\int_0^{\infty} \frac{\sin \alpha x \sin \beta x}{x}\, dx = \frac{1}{2} \ln \left|\frac{\alpha + \beta}{\alpha - \beta}\right|$.

185. $$\int_0^{\infty} \frac{\sin \alpha x \cos \beta x}{x}\, dx = \begin{cases} 0 & \text{for } |\alpha| < |\beta|, \\ \frac{\pi}{4} & \text{for } |\alpha| = |\beta|. \\ \frac{\pi}{2} \operatorname{sign} \alpha & \text{for } |\alpha| > |\beta|. \end{cases}$$

186. $\int_0^\infty \frac{\sin^3 \alpha x}{x}\, dx = \frac{\pi}{4} \operatorname{sign} \alpha.$

187. $\int_0^\infty \frac{\sin^4 \alpha x - \sin^4 \beta x}{x}\, dx = \frac{3}{8} \ln \left| \frac{\alpha}{\beta} \right|.$

188. $\int_0^\infty \frac{\sin^2 \alpha x}{x^2}\, dx = \frac{\pi}{2} |\alpha|.$

189. $\int_0^\infty \frac{\sin^3 \alpha x}{x^3}\, dx = \frac{3}{8} \pi\alpha |\alpha|.$

190. $\int_0^\infty \frac{\sin^4 x}{x^2}\, dx = \frac{\pi}{4}.$

191. $\int_0^\infty \frac{\sin^2 x}{1 + x^2}\, dx = \frac{\pi}{4} (1 - e^{-2}).$

192. $\int_0^\infty \frac{x \sin \alpha x}{1 + x^2}\, dx = \frac{\pi}{2} \operatorname{sign} \alpha\, e^{-|\alpha|}.$

193. $\int_0^\infty \frac{\cos \alpha x}{1 + x^2}\, dx = \frac{\pi}{2} e^{-|\alpha|}.$

194. $\int_0^\infty \frac{\cos \alpha x}{(1 + x^2)^2}\, dx = \frac{\pi(1 + |\alpha|)}{4} e^{-|\alpha|}.$

195. $\int_0^\infty \frac{\sin \alpha x}{x^m}\, dx = \frac{\pi \alpha^{m-1}}{2\Gamma(m) \sin \frac{m\pi}{2}} \qquad (0 < m < 2, \quad \alpha > 0).$

196. $\int_0^\infty \frac{\cos \alpha x}{x^m}\, dx = \frac{\pi \alpha^{m-1}}{2\Gamma(m) \cos \frac{m\pi}{2}} \qquad (0 < m < 1, \quad \alpha > 0).$

197. $\int_0^\pi \frac{\sin^{n-1} x}{(1 + k \cos x)^n}\, dx = \frac{2^{n-1}}{(1 - k^2)^{n/2}} B\left(\frac{n}{2}, \frac{n}{2}\right)$

$(n > 0,\ 0 < |k| < 1).$

198. $\int_0^\infty e^{-\alpha x} \sin \beta x\, dx = \frac{\beta}{\alpha^2 + \beta^2}.$

199. $\int_0^\infty e^{-\alpha x} \cos \beta x\, dx = \frac{\alpha}{\alpha^2 + \beta^2}.$

200. $\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \sin mx\, dx = \arctan \frac{\beta}{m} - \arctan \frac{\alpha}{m}$

$(\alpha > 0, \quad \beta > 0, \quad m \neq 0).$

201. $\displaystyle\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos mx \, dx = \frac{1}{2} \ln \frac{\beta^2 + m^2}{\alpha^2 + m^2} \qquad (\alpha > 0, \quad \beta > 0).$

202. $\displaystyle\int_0^\infty e^{-kx} \frac{\sin \alpha x}{x} \frac{\sin \beta x}{x} dx$

$$= \frac{\alpha + \beta}{2} \arctan \frac{\alpha + \beta}{k} - \frac{\alpha - \beta}{2} \arctan \frac{\alpha - \beta}{k} + \frac{k}{4} \ln \frac{k^2 + (\alpha - \beta)^2}{k^2 + (\alpha + \beta)^2}$$

$(k \geqslant 0, \quad \alpha > 0, \quad \beta > 0).$

203. $\displaystyle\int_0^\infty e^{-\alpha x^2} \cos \beta x \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} e^{-\beta^2/4\alpha} \qquad (\alpha > 0).$

204. $\displaystyle\int_0^\infty x e^{-\alpha x^2} \sin \beta x \, dx = \frac{\beta \sqrt{\pi}}{4\alpha \sqrt{\alpha}} e^{-\beta^2/4\alpha} \qquad (\alpha > 0).$

205. $\displaystyle\int_0^\infty x^{2n} e^{-x^2} \cos 2bx \, dx = (-1)^n \frac{\sqrt{\pi}}{2^{2n+1}} \frac{d^{2n}(e^{-b^2})}{db^{2n}}$

(n is a natural number).

206. $\displaystyle\int_0^\infty \frac{e^{-\alpha x^2} - \cos \beta x}{x^2} dx = \pi \frac{|\beta|}{2} - \sqrt{\pi \alpha} \qquad (\alpha > 0).$

207. $\displaystyle\int_0^{\pi/2} \ln (a^2 \sin^2 x + b^2 \cos^2 x) \, dx = \pi \ln \frac{a + b}{2}.$

208. $\displaystyle\int_0^\pi \ln (1 - 2a \cos x + a^2) \, dx = \begin{cases} 0 & \text{for} \quad |a| \leqslant 1; \\ \pi \ln a^2 & \text{for} \quad |a| > 1. \end{cases}$

209. $\displaystyle\int_0^{\pi/2} \ln \frac{1 + a \cos x}{1 - a \cos x} \frac{dx}{\cos x} = \pi \arcsin a \qquad (|a| < 1).$

210. $\displaystyle\int_0^{\pi/2} \frac{\arctan (\alpha \tan x)}{\tan x} dx = \frac{\pi}{2} \operatorname{sign} \alpha \ln (1 + |\alpha|).$

211. $\displaystyle\int_0^\infty \frac{\arctan \alpha x - \arctan \beta x}{x} dx = \frac{\pi}{2} \ln \frac{\alpha}{\beta} \qquad (\alpha > 0, \quad \beta > 0).$

212. $\displaystyle\int_0^\infty \frac{\arctan \alpha x \arctan \beta x}{x^2} dx = \frac{\pi}{2} \ln \frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^\alpha \beta^\beta} \qquad (\alpha > 0, \quad \beta > 0).$

213. $\displaystyle\int_0^\infty \frac{\arctan \alpha x}{x^2 \sqrt{x^2 - 1}} dx = \frac{\pi}{2} \operatorname{sign} \alpha \left(1 + |\alpha| - \sqrt{1 + \alpha^2}\right).$

214. $\displaystyle\int_0^1 \ln \Gamma(x) \, dx = \ln \sqrt{2\pi}.$

215. $\int_a^{a+1} \ln \Gamma(x)\,dx = \ln \sqrt{2\pi} + a(\ln a - 1) \qquad (a > 0).$

216. $\int_0^1 \ln \Gamma(x) \sin \pi x\,dx = \frac{1}{\pi}\left(1 + \ln \frac{\pi}{2}\right).$

217. $\int_0^1 \ln \Gamma(x) \cos(2n\pi x)\,dx = \frac{1}{4n} \qquad$ (n is a natural number).

218. $\int_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \cos(\lambda t \sin\alpha)\,dt = \frac{\Gamma(x)}{\lambda^x} \cos \alpha x;$

219. $\int_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \sin(\lambda t \sin\alpha)\,dt = \frac{\Gamma(x)}{\lambda^x} \sin \alpha x$

Euler formulae

$$\left(\lambda > 0,\ x > 0,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right).$$

Table F. Multiple Integrals

220. $\int_0^x dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-1}} f(x_n)\,dx_n = \int_0^x f(u) \frac{(x-u)^{n-1}}{(n-1)!}\,du.$

221. $\int_0^x x_1\,dx_1 \int_0^{x_1} x_2\,dx_2 \ldots \int_0^{x_{n-1}} x_n f(x_n)\,dx_n$

$$= \frac{1}{2^n n!} \int_0^x (x^2 - u^2)^n f(u)\,du.$$

222. $\int_0^1 \int_0^1 \ldots \int_0^1 (x_1^2 + x_2^2 + \cdots + x_n^2)\,dx_1\,dx_2 \ldots dx_n = \frac{n}{3}.$

223. $\int_0^1 \int_0^1 \ldots \int_0^1 (x_1 + x_2 + \cdots + x_n)^2\,dx_1\,dx_2 \cdots dx_n = \frac{n(3n+1)}{12}.$

224. $\int\int \ldots \int_\Omega dx_1\,dx_2 \ldots dx_n = \frac{a^n}{n!},$

where Ω is a region defined by the inequalities $x_i \geqslant 0$ $(i = 1, 2, \ldots, n)$ and $x_1 + x_2 + \cdots + x_n \leqslant a$.

225. $\int\int \ldots \int_\Omega x_n^2\,dx_1\,dx_2 \ldots dx_n = \frac{\pi^{\frac{n-1}{2}} a^{n-1} h^3}{12\Gamma\left(\frac{n+1}{2}\right)},$

where the region Ω is defined by the inequalities

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leqslant a^2, \quad -\frac{h}{2} \leqslant x_n \leqslant \frac{h}{2}.$$

226. $$\int\int\cdots\int_{\Omega}\frac{dx_1\,dx_2\ldots dx_n}{\sqrt{1-x_1^2-x_2^2-\cdots-x_n^2}}=\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)},$$

where Ω is a region defined by the inequality:

$$x_1^2+x_2^2+\cdots+x_n^2\leqslant 1.$$

If $f(u)$ is a continuous function, then for $n\geqslant 2$:

227. $$\int\int\cdots\int_{\Omega}f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})\,dx_1\,dx_2\ldots dx_n$$
$$=2\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^R f(r)r^{n-1}\,dr,$$

where Ω is a region defined by the inequality

$$x_1^2+x_2^2+\cdots+x_n^2\leqslant R^2.$$

228. $$\int\int\cdots\int_{\Omega}x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n+1)}\qquad(p_1,p_2,\ldots,p_n>0),$$

where Ω is a region defined by the inequalities

$$x_1,x_2,\ldots,x_n\geqslant 0,\quad x_1+x_2+\cdots+x_n\geqslant 1\quad\textit{(Dirichlet formula)}.$$

229. $$\int\int\cdots\int_{\Omega}f(x_1+x_2+\cdots+x_n)\,x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}dx_1\,dx_2\ldots dx_n$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 f(u)\,u^{p_1+p_2+\cdots+p_n-1}\,du$$

($p_1,p_2,\ldots,p_n>0$), where $f(u)$ is a continuous function and the integral on the right converges absolutely, and Ω is a region, defined by the inequalities $x_1,x_2,\ldots,x_n\geqslant 0$, $x_1+x_2+\cdots+x_n\leqslant 1$ *(Liouville's formula)*.

If $f=f(x_1,x_2,\ldots,x_n)$ is a continuous function in the region $0\leqslant x_i\leqslant x$ ($i=1,2,\ldots,n$) then

230. $$\int_0^x dx_1\int_0^{x_1}dx_2\ldots\int_0^{x_{n-1}}f\,dx_n$$
$$=\int_0^x dx_n\int_{x_n}^x dx_{n-1}\ldots\int_{x_2}^x f\,dx_1\quad(n\geqslant 2).$$

If $K(x, y)$ is a continuous function in the region $R[a \leqslant x \leqslant b;\ a \leqslant y \leqslant b]$ and

$$K_n(x, y) = \int_a^b \int_a^b \cdots \int_a^b K(x, t_1)\, K(t_1, t_2) \ldots K(t_n, y)\, dt_1\, dt_2 \ldots dt_n,$$

then

231. $$K_{n+m}(x, y) = \int_a^b K_n(x, t)\, K_m(t, y)\, dt.$$

If $\sum\limits_{i,j=1}^{n} a_{ij} x_i x_j$ $(a_{ij} = a_{ji})$ is a positively defined form, then

232. $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\left\{\sum\limits_{i,j=1}^{n} a_{ij}x_i x_j + 2 \sum\limits_{i=1}^{n} b_i x_i + c\right\}} dx_1\, dx_2 \ldots dx_n$$

$$= \sqrt{\frac{\pi^n}{|\delta|}}\, e^{-\Delta/\delta},$$

where $\delta = |a_{ij}|$, $\Delta = \begin{vmatrix} a_{ij} & b_i \\ b_j & c \end{vmatrix}$ is a bounded determinant.

The volume of an n-dimensional parallelepiped, bounded by the planes

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \pm h_i \qquad (i = 1, 2, \ldots, n),$$

if $\Delta = |a_{ij}| \neq 0$, equals

233. $$V_n = \frac{2^n h_1 h_2 \ldots h_n}{|\Delta|}.$$

The volume of an n-dimensional pyramid, bounded by the planes

$$x_i \geqslant 0 \quad (i = 1, 2, \ldots, n),$$

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1 \quad (a_i > 0, \quad i = 1, 2, \ldots, n),$$

equals

234. $$V_n = \frac{a_1 a_2 \ldots a_n}{n!}.$$

The volume of an n-dimensional cone bounded by the surfaces

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_{n-1}^2}{a_{n-1}^2} = \frac{x_n^2}{a_n^2}, \quad x_n = a_n,$$

equals

235. $$V_n = \frac{\pi^{\frac{n-1}{2}}}{n\Gamma\left(\frac{n+1}{2}\right)}\, a_1 a_2 \ldots a_n.$$

The volume of an n-dimensional sphere $x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant a^2$ equals

236. $V_n = \dfrac{\pi^{n/2} a^n}{\Gamma\left(\dfrac{n}{2} + 1\right)}$,

in particular,

237. $V_{2m} = \dfrac{\pi^m}{m!} a^{2m}$.

238. $V_{2m+1} = \dfrac{2(2\pi)^m a^{2m+1}}{(2m + 1)!!}$.

The area of the surface of an n-dimensional sphere, $x_1^2 + x_2^2 + \cdots + x_n^2 = a^2$, equals

239. $S_n = 2 \dfrac{\pi^{n/2} a^{n-1}}{\Gamma\left(\dfrac{n}{2}\right)}$.

For the calculation of the volume V_n and the area S_n of the surface of an n-dimensional sphere, the following formulae can also be used:

240. $V_n = \dfrac{\pi^\nu}{\nu!} R^n$ for $n = 2\nu$ $(\nu = 1, 2, \ldots)$.

241. $V_n = \dfrac{\pi^\nu \nu!}{n!} (2R)^n$ for $n = 2\nu + 1$ $(\nu = 1, 2, \ldots)$.

242. $S_n = \dfrac{n}{R} V_n$,

where R is the radius of the sphere.
In particular

243. $V_2 = \pi R^2$, $S_2 = 2\pi R$;

244. $V_3 = \dfrac{4}{3} \pi R^3$, $S_3 = 4\pi R^2$;

245. $V_4 = \dfrac{\pi^2}{2} R^4$, $S_4 = 2\pi^2 R^3$;

246. $V_5 = \dfrac{8}{15} \pi^2 R^5$, $S_5 = \dfrac{8}{3} \pi^2 R^4$;

247. $V_6 = \dfrac{\pi^3}{6} R^6$, $S_6 = \pi^3 R^5$;

248. $V_7 = \dfrac{16}{105} \pi^3 R^7$, $S_7 = \dfrac{16}{15} \pi^3 R^6$;

249. $V_8 = \frac{\pi^4}{24} R^8, \qquad S_8 = \frac{\pi^4}{3} R^7;$

250. $V_9 = \frac{32}{945} \pi^4 R^9, \qquad S_9 = \frac{32}{105} \pi^4 R^8;$

251. $V_{10} = \frac{\pi^5}{120} R^{10}, \qquad S_{10} = \frac{\pi^5}{12} R^9.$

For $n \gg 1$ there occur the asymptotic formulae:

252. $V_n \approx \frac{1}{\sqrt{\pi n}} \left(\frac{2\pi e}{n}\right)^{n/2} R^n,$

253. $S_n \approx \sqrt{\frac{n}{\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} R^{n-1}.$

The potential u of a homogeneous sphere of radius R and density δ_0 equals

254. $u = \frac{\delta_0^2}{2} \int\int\int\int\int\int_{\Omega} \frac{dx_1\,dy_1\,dz_1\,dx_2\,dy_2\,dz_2}{r_{1,2}} = \frac{16}{15}\pi^2\delta_0^2 R^5,$

where $r_{1,2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, and Ω is a region defined by the inequalities

$$x_1^2 + y_1^2 + z_1^2 \leqslant R^2, \quad x_2^2 + y_2^2 + z_2^2 \leqslant R^2.$$

4. Special Functions defined by Integrals

Special functions defined by integrals of elementary functions, which, however, are not expressible in terms of elementary functions, are given below together with the corresponding numerical tables. For more detailed data on these non-elementary functions, see volume 69 of this series and refs. 5, 16, 19, 22, 24, 25, 26, 27, 29, 43.

1°. Elliptic Integrals

Elliptic integral of the first kind:

$$F(k, \varphi) = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} = \int_0^{\varphi} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}.$$

TABLE 1. ELLIPTIC INTEGRALS OF THE FIRST KIND $F(k, \varphi)$, $k = \sin \alpha$

k^2	0·00000	0·03015	0·11698	0·25000	0·41318	0·58682	0·75000	0·88302	0·96985	1·00000
φ \ α	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
10°	0·17453	0·17416	0·17464	0·17475	0·17490	0·17505	0·17520	0·17532	0·17540	0·17543
20°	34907	34927	34988	35082	35326	35199	35447	35548	35615	35638
30°	52360	52428	52628	52943	53343	53787	54223	54593	54843	54931
40°	69813	69969	70429	71165	72126	73231	74358	75352	0·76043	0·76291
50°	0·87266	0·87556	0·88416	0·89825	0·91725	0·94008	0·96465	0·98762	1·00444	1·01068
60°	1·04720	1·05188	1·06597	1·08955	1·12256	1·16432	1·21260	1·26186	30135	31696
70°	22173	22861	24953	28530	33723	40677	49441	1·59591	1·69181	1·73542
80°	39626	40565	43442	48455	55973	66597	1·81253	2·01193	2·26527	2·43625
90°	1·57080	1·58284	1·62003	1·68575	1·78677	1·93558	2·15652	2·50455	3·15339	∞

TABLE 2. ELLIPTIC INTEGRALS OF THE SECOND KIND $E(k, \varphi)$, $k = \sin \alpha$

k^2	0·00000	0·03015	0·11698	0·25000	0·41318	0·58682	0·75000	0·88302	0·96985	1·00000
φ \ α	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
10°	0·17453	0·17451	0·17443	0·17431	0·17417	0·17401	0·17387	0·17375	0·17367	0·17365
20°	34907	34886	34825	34733	34619	34496	34381	34286	34224	34202
30°	52360	52292	52094	51788	51409	51000	50609	50287	50074	50000
40°	69813	69658	69207	68506	67628	66671	65746	64974	64459	64279
50°	0·87266	0·86979	0·86142	0·84832	83173	81338	79538	78007	76971	76604
60°	1·04720	1·04255	1·02897	1·00756	0·98013	0·94930	0·91839	89144	87276	86603
70°	22173	21491	19493	16318	1·12205	1·07500	1·02664	0·98298	0·95144	93969
80°	39626	38698	35968	31606	25897	19255	12249	1·05648	1·00543	0·98481
90°	1·57080	1·55889	1·52380	1·46746	1·39314	1·30554	1·21106	1·11838	1·04011	1·00000

TABLE 3. COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS, K AND E, $k = \sin\alpha$

k^2	α	K	E	k^2	α	K	E
0·00000	0°	1·57080	0·57080	0·51745	46°	1·86915	1·34181
00030	1°	57092	57068	53488	47°	88481	33287
00122	2°	57127	57032	55226	48°	90108	32384
00274	3°	57187	56972	56959	49°	91800	31473
00487	4°	57271	56888	58682	50°	93558	30554
00760	5°	57379	56781	60396	51°	95386	29628
01093	6°	57511	56650	62096	52°	97288	28695
01485	7°	57668	56495	63782	53°	1·99267	27757
01937	8°	57849	56316	65451	54°	2·01327	26815
02447	9°	58054	56114	67101	55°	03472	25868
03015	10°	58284	55889	68730	56°	05706	24918
03641	11°	58539	55640	70337	57°	08036	23966
04323	12°	58820	55368	71919	58°	10466	23013
05060	13°	59125	55073	73474	59°	13002	22059
05853	14°	59457	54755	75000	60°	15652	21106
06699	15°	59814	54415	76496	61°	18421	20154
07598	16°	60198	54052	77960	62°	21319	19205
08548	17°	60608	53667	79389	63°	24355	18259
09549	18°	61045	53260	80783	64°	27538	17318
10599	19°	61510	52831	82139	65°	30879	16383
11698	20°	62003	52380	83457	66°	34390	15455
12843	21°	62523	51908	84733	67°	38087	14535
14033	22°	63073	51415	85967	68°	41984	13624
15267	23°	63652	50901	87157	69°	46100	12725
16543	24°	64260	50366	88302	70°	50455	11838
17861	25°	64900	49811	89401	71°	55073	10964
19217	26°	65570	49237	90451	72°	59982	10106
20611	27°	66272	48643	91452	73°	65214	09265
22040	28°	67006	48029	92402	74°	70807	08443
23504	29°	67773	47397	93301	75°	76806	07641
25000	30°	68575	46746	94147	76°	83267	06861
26526	31°	69411	46077	94940	77°	90256	06106
28081	32°	70284	45391	95677	78°	2·97857	05378
29663	33°	71192	44687	96359	79°	3·06173	04679
31270	34°	72139	43966	96985	80°	15339	04011
32899	35°	73125	43229	97553	81°	25530	03379
34549	36°	74150	42476	98063	82°	36987	02784
36218	37°	75217	41707	98515	83°	50042	02231
37904	38°	76326	40924	98907	84°	65186	01724
39604	39°	77479	40126	99240	85°	3·83174	01266
41318	40°	78677	39314	99513	86°	4·05276	00865
43041	41°	79922	38486	99726	87°	33865	00526
44774	42°	81216	37650	99878	88°	4·74272	00258
46512	43°	82560	36800	0·99970	89°	5·43491	00075
48255	44°	83957	35938	1·00000	90°	∞	1·00000
0·50000	45°	1·85407	1·35064				

Elliptic integral of the second kind:

$$E(k, \varphi) = \int_0^{\sin \varphi} \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}}\, dx = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi .$$

Complete elliptic integral of the first kind:

$$K = F\left(k, \frac{\pi}{2}\right) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

$$= \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} .$$

Complete elliptic integral of the second kind:

$$E = E\left(k, \frac{\pi}{2}\right) = \int_0^1 \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}}\, dx = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi .$$

2°. Integral Functions

Integral sine:

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!\,(2k-1)} .$$

Integral cosine:

$$\operatorname{Ci}(x) = -\int_x^{\infty} \frac{\cos t}{t}\, dt = C + \ln x + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!\, 2k} \quad (x > 0),$$

where $C = 0{\cdot}57721566$ – Euler's constant.

Integral exponential function:

$$-\operatorname{Ei}(-x) = \int_x^{\infty} \frac{e^{-t}}{t}\, dt = -C - \ln x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k!\, k} \quad (x > 0),$$

$$\operatorname{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt = C + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{k!\, k} \quad (x > 0).$$

The latter integral is understood in the sense of the principal value, it is sometimes denoted by the symbol $\overline{\mathrm{Ei}}(x)$.

Integral logarithm:

$$\mathrm{li}\,(x) = \int_0^x \frac{dt}{\ln t} = C + \ln(-\ln x) + \sum_{k=2}^{\infty} \frac{(\ln x)^k}{k!\,k} \quad (0 < x < 1).$$

There exist relationships permitting the change from the values of the integral exponential function to the values of the integral logarithm and back again

$$\mathrm{Ei}\,(\ln x) = \mathrm{li}\,(x) \quad (0 < x < 1),$$

$$\mathrm{li}\,(e^x) = \mathrm{Ei}\,(x) \quad (x < 0).$$

TABLE 4. INTEGRAL FUNCTIONS $\mathrm{Si}\,(x)$, $\mathrm{Ci}(x)$, $\overline{\mathrm{Ei}}(x)$, $-\mathrm{Ei}\,(-x)$

x	$\mathrm{Si}(x)$	$\mathrm{Ci}(x)$	$\overline{\mathrm{Ei}}(x)$	$-\mathrm{Ei}(-x)$
0·00	0·0000000	$-\infty$	$-\infty$	∞
01	0099999	−4·02798	−4·01793	4·03793
02	0199996	−3·33491	−3·31471	3·35471
03	029998	−2·92957	−2·89912	2·95912
04	039996	−2·64206	−2·60126	2·68126
05	049993	−2·41914	−2·36788	2·46790
06	059988	23709	17528	29531
07	069981	−2·08327	−2·01080	15084
08	079972	−1·95011	−1·86688	2·02694
09	089960	83275	73866	1·91874
10	099944	72787	62281	82292
11	10993	63308	51696	73711
12	11990	54665	41935	65954
13	12988	46723	32866	58890
14	13985	39379	24384	52415
15	14981	32552	16409	46446
16	15977	26176	08873	40919
17	16973	20196	−1·01723	35778
18	17968	14567	−0·94915	30980
19	18962	09253	88410	26486
20	19956	−1·04221	82176	22270
21	20949	−0·99444	76187	18290
22	21941	94899	70420	14538
23	22933	90566	64853	10988
24	23923	86427	59470	07624
0·25	0·24913	−0·82466	−0·54254	1·04428

TABLE 4 *(contd.)*

x	Si(x)	Ci(x)	$\overline{\text{Ei}}(x)$	$-\text{Ei}(-x)$
0·26	0·25903	−0·78671	−0·49193	1·01389
27	26891	75029	44274	0·98493
28	27878	71529	39486	95731
29	28865	68161	34820	93092
30	29850	64917	30267	90568
31	30835	61790	25819	88151
32	31819	58771	21468	85834
33	32801	55855	17210	83610
34	33782	53036	13036	81475
35	34763	50308	08943	79422
36	35742	47666	04926	77446
37	36720	45107	−0·00979	75544
38	37696	42625	+0·02901	73711
39	38672	40218	06718	71944
40	39646	37881	10477	70238
41	40619	35611	14179	68591
42	41591	33406	17828	67000
43	42561	31262	21427	65461
44	43530	29178	24979	63973
45	44497	27149	28486	62533
46	45463	25175	31950	61139
47	46427	23253	35374	59788
48	47390	21380	38759	58478
49	48351	19556	42108	57209
50	49311	17778	45422	55977
51	50269	16045	48703	54782
52	51225	14355	51953	53622
53	52180	12707	55173	52495
54	53133	11099	58365	51400
55	54084	095300	61529	50336
56	55033	079986	64668	49302
57	55981	065037	67782	48296
58	56927	050442	70873	47317
59	57871	036190	73941	46365
60	58813	022271	76988	45438
61	59753	−0·0086752	80015	44535
62	60691	+0·0046060	83023	43656
63	61627	017582	86012	42800
64	62561	030260	88984	41965
65	63494	042650	91939	41152
66	64424	054758	94878	40359
67	65351	066591	0·97802	39585
68	66277	078158	1·00712	38831
0·69	0·67201	0 089463	1·03608	0·38095

TABLE 4 *(contd.)*

x	$\mathrm{Si}(x)$	$\mathrm{Ci}(x)$	$\overline{\mathrm{Ei}}(x)$	$-\mathrm{Ei}(-x)$
0·70	0·68122	0·10051	1·06491	0·37377
71	69041	11132	09362	36676
72	69958	12188	12220	35992
73	70873	13220	15068	35324
74	71785	14230	17906	34671
75	72695	15216	20733	34034
76	73603	16181	23551	33412
77	74508	17124	26360	32803
78	75411	18046	29161	32209
79	76312	18947	31954	31628
80	77210	19828	34740	31060
81	78105	20689	37518	30504
82	78998	21530	40290	29961
83	79888	22353	43056	29430
84	80776	23157	45816	28910
85	81661	23942	48571	28402
86	82544	24710	51322	27905
87	83424	25460	54067	27418
88	84301	26192	56809	26941
89	85175	26908	59547	26475
90	86047	27607	62281	26018
91	86916	28289	65013	25571
92	87782	28956	67741	25134
93	88646	29606	70468	24705
94	89506	30242	73192	24285
95	90364	30861	75915	23874
96	91219	31466	78636	23471
97	92070	32056	81356	23076
98	92919	32632	84075	22689
0·99	93765	33193	86794	22310
1·0	0·94608	33740	1·89512	21938
1·1	1·02869	38487	2·16738	18600
1·2	10805	42046	2·44209	15841
1·3	18396	44574	2·72140	13545
1·4	25623	46201	3·00721	11622
1·5	32468	47036	3·30129	10002
1·6	38918	47173	3·60532	086308
1·7	44959	46697	3·92096	074655
1·8	50582	45681	4·24987	064713
1·9	55778	44194	4·59371	056204
2·0	60541	42298	4·95423	048901
2·1	64870	40051	5·33324	042614
2·2	68762	37508	5·73262	037191
2·3	1 72221	0·34718	6·15438	0·032502

TABLE 4 *(contd.)*

x	Si(x)	Ci(x)	$\overline{\mathrm{Ei}}(x)$	$-\mathrm{Ei}(-x)$
2·4	1·75249	0·31730	6·60067	0·028440
2·5	77852	28587	7·07377	024915
2·6	80039	25334	7·57612	021850
2·7	81821	22008	8·11035	019182
2·8	83210	18649	8·67930	016855
2·9	84219	15290	9·28602	014824
3·0	84865	11963	9·93383	013048
3·1	85166	086992	10·6263	011494
3·2	85140	055257	11·3673	010133
3·3	84808	+0·024678	12·1610	$0^{2}8939$†
3·4	84191	−0·0045181	13·0121	$0^{2}7891$
3·5	83313	032128	13·9254	$0^{2}6970$
3·6	82195	057974	14·9063	$0^{2}6160$
3·7	80862	081901	15·9606	$0^{2}5448$
3·8	79339	10378	17·0948	$0^{2}4820$
3·9	77650	12350	18·3157	$0^{2}4267$
4.0	75820	14098	19·6309	$0^{2}3779$
4·1	73874	15617	21·0485	$0^{2}3349$
4·2	71837	16901	22·5774	$0^{2}2969$
4·3	69732	17951	24·2274	$0^{2}2633$
4·4	67583	18766	26·0090	$0^{2}2336$
4·5	65414	19349	27·9337	$0^{2}2073$
4·6	63246	19705	30·0141	$0^{2}1841$
4·7	61101	19839	32·2639	$0^{2}1635$
4·8	58998	19760	34·6979	$0^{2}1453$
4·9	56956	19478	37·3325	$0^{2}1291$
5·0	54993	19003	40·1853	$0^{2}1148$
6·0	42469	−0·068057	85·9898	$0^{3}3601$
7·0	45460	+0·076695	191·505	$0^{3}1155$
8·0	57419	+0·122434	440·380	$0^{4}3767$
9·0	66504	+0·055348	1037·88	$0^{4}1245$
10	65835	−0·045456	2492·23	$0^{5}4157$
11	57831	−0·089561	6071·41	$0^{5}1400$
12	50497	−0·049780	14959·5	$0^{6}4751$
13	49936	+0·026764	37197·7	$0^{6}1622$
14	55621	+0·069396	93192·5	$0^{7}5566$
15	61819	+0·046279	234956	$0·0^{7}1919$
16	63130	−0·014200		
17	59014	−0·055243		
18	53661	−0·043475		
19	1·51863	+0·0051504		

† The figure printed in small type at the top denotes the number of zeros after the decimal point.

TABLE 4 *(contd.)*

x	Si(x)	Ci(x)	x	Si(x)	Ci(x)
20	1·54824	+0·044420	140	1·5722	+0·007011
25	53148	−0·006849	150	5662	−0·004800
30	56676	−0·033032	160	5769	+0·001409
35	59692	−0·011480	170	5653	+0·002010
40	58699	+0·019020	180	5741	−0·004432
45	55872	+0·018632	190	5704	+0·005250
50	55162	−0·005628	200	5684	−0·004378
55	57072	−0·018173	300	5709	−0·003332
60	58675	−0·004813	400	5721	−0·002124
65	57925	+0·012847	500	5726	−0·0009320
70	56159	+0·010922	600	5725	+0·0000764
75	55858	−0·005332	700	5720	+0·0007788
80	57233	−0·012402	800	5714	+0·001118
85	58240	−0·001935	900	5707	+0·001109
90	57566	+0·009986	10^3	5702	+0·000826
95	56304	+0·007110	10^4	5709	−0·0000306
100	56223	−0·005149	10^5	5708	+0·0000004
110	5799	−0·000320	10^6	5708	−0·0000004
120	5640	+0·004781	10^7	1·5708	+0·0000000
130	1·5737	−0.007132	∞	$\frac{1}{2}\pi$	+0·0000000

3°. Integrals of Probability

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(k-1)!\,(2k-1)},$$

$$\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2}\, dt.$$

Relationships between them:

$$\operatorname{erf} x = \Phi\left(x\sqrt{2}\right),$$

$$\Phi(x) = \operatorname{erf} \frac{x}{\sqrt{2}}.$$

Derivatives of the integral of probability:

$$\frac{d}{dx}(\operatorname{erf} x) = \frac{2}{\sqrt{\pi}} e^{-x^2},$$

$$\frac{d}{dx}\Phi(x) = \sqrt{\frac{2}{\pi}}\, e^{-x^2/2} = 2\varphi(x).$$

TABLE 5. INTEGRAL OF PROBABILITY

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

x	erf x	x	erf x	x	erf x	x	erf x
0·00	0·00000	0·38	0·40901	0·75	0·71116	1·13	0·88997
01	01128	39	41874	76	71754	14	89308
02	02256	40	42839	77	72382	15	89612
03	03384	41	43797	78	73001	16	89910
04	04511	42	44747	79	73610	17	90200
05	05637	43	45689	80	74210	18	90484
06	06762	44	46623	81	74800	19	90761
07	07886	45	47548	82	75381	20	91031
08	09008	46	48466	83	75952	21	91296
09	10128	47	49375	84	76514	22	91553
10	11246	48	50275	85	77067	23	91805
11	12362	49	51167	86	77610	24	92051
12	13476	50	52050	87	78144	25	92290
13	14587	51	52924	88	78669	26	92524
14	15695	52	53790	89	79184	27	92751
15	16800	53	54646	90	79691	28	92973
16	17901	54	55494	91	80188	29	93190
17	18999	55	56332	92	80677	30	93401
18	20094	56	57162	93	81156	31	93606
19	21184	57	57982	94	81627	32	93807
20	22270	58	58792	95	82089	33	94002
21	23352	59	59594	96	82542	34	94191
22	24430	60	60386	97	82987	35	94376
23	25502	61	61168	98	83423	36	94556
24	26570	62	61941	0·99	83851	37	94731
25	27633	63	62705	1·00	84270	38	94902
26	28690	64	63459	01	84681	39	95067
27	29742	65	64203	02	85084	40	95229
28	30788	66	64938	03	85478	41	95385
29	31828	67	65663	04	85865	42	95538
30	32863	68	66379	05	86244	43	95686
31	33891	69	67084	06	86614	44	95830
32	34913	70	67780	07	86977	45	95970
33	35928	71	68467	08	87333	46	96105
34	36936	72	69143	09	87680	47	96237
35	37938	73	69810	10	88021	48	96365
36	38933	0·74	0·70468	11	88353	1·49	0·96490
0·37	0·39921			1·12	0·88679		

TABLE 5 (contd.)

x	erf x	x	erf x	x	erf x	x	erf x
1·50	0·96611	1·68	0·98249	1·85	0·99111	2·03	0·99591
51	96728	69	98315	86	99147	04	99609
52	96841			87	99182		
53	96952	70	98379	88	99216	05	99626
54	97059	71	98441	89	99248	06	99642
		72	98500			07	99658
55	97162	73	98558	90	99279	08	99673
56	97263	74	98613	91	99309	09	99688
57	97360			92	99338		
58	97455	75	98667	93	99366	10	99702
59	97546	76	98719	94	99392	11	99716
		77	98769			12	99728
60	97635	78	98817	95	99418	13	99741
61	97721	79	98864	96	99443	14	99752
62	97804			97	99466		
63	97884	80	98909	98	99489	15	99764
64	97962	81	98952	1·99	99511	16	99775
65	98038	82	98994	2·00	99532	17	99785
66	98110	83	99035	01	99552	18	99795
1·67	0·98181	1·84	0·99074	2·02	0·99572	2·19	0·99805

TABLE 6. INTEGRAL OF PROBABILITY

$$\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2}\, dt$$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0·00	0·00000	0·15	11924	0·30	0·23582	0·45	0·34729
01	00798	16	12712	31	24344	46	35448
02	01596	17	13499	32	25103	47	36164
03	02393	18	14285	33	25860	48	36877
04	03191	19	15069	34	26614	49	37587
05	03988	20	15852	35	27366	50	38292
06	04784	21	16633	36	28115	51	38995
07	05581	22	17413	37	28862	52	39694
08	06376	23	18191	38	29605	53	40389
09	07171	24	18967	39	30346	54	41080
10	07966	25	19741	40	31084	55	41768
11	08759	26	20514	41	31819	56	42452
12	09552	27	21284	42	32552	57	43132
13	10343	28	22052	43	33280	58	43809
14	0·11134	0·29	22818	0·44	0·34006	0·59	0·44481

TABLE 6 *(contd.)*

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0·60	0·45149	1·00	0·68269	1·40	0·83849	1·80	0·92814
61	45814	01	68750	41	84146	81	92970
62	46474	02	69227	42	84439	82	93124
63	47131	03	69699	43	84728	83	93275
64	47783	04	70166	44	85013	84	93423
65	48431	05	70628	45	85294	85	93569
66	49075	06	71086	46	85571	86	93711
67	49714	07	71538	47	85844	87	93852
68	50350	08	71986	48	86113	88	93989
69	50981	09	72429	49	86378	89	94124
70	51607	10	72867	50	86639	90	94257
71	52230	11	73300	51	86896	91	94387
72	52848	12	73729	52	87149	92	94514
73	53461	13	74152	53	87398	93	94639
74	54070	14	74571	54	87644	94	94762
75	54675	15	74986	55	87886	95	94882
76	55275	16	75395	56	88124	96	95000
77	55870	17	75800	57	88358	97	95116
78	56461	18	76200	58	88589	98	95230
79	57047	19	76595	59	88817	1·99	95341
80	57629	20	76986	60	89040	2·00	95450
81	58206	21	77372	61	89260	01	95557
82	58778	22	77754	62	89477	02	95662
83	59346	23	78130	63	89960	03	95764
84	59909	24	78502	64	89899	04	95865
85	60468	25	78870	65	90106	05	95964
86	61021	26	79233	66	90309	06	96060
87	61570	27	79592	67	90508	07	96155
88	62114	28	79945	68	90704	08	96247
89	62653	29	80295	69	90897	09	96338
90	63188	30	80640	70	91087	10	96427
91	63718	31	80980	71	91273	11	96514
92	64243	32	81316	72	91457	12	96599
93	64763	33	81648	73	91637	13	96683
94	65278	34	81975	74	91814	14	96765
95	65789	35	82298	75	91988	15	96844
96	66294	36	82617	76	92159	16	96923
97	66795	37	82931	77	92327	17	96999
98	67291	38	83241	78	92492	18	97074
0·99	0·67783	1·39	0·83547	1·79	0·92655	2·19	0·97148

TABLE 6 *(contd.)*

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
2·20	0·97219	2·55	0·98923	2·90	0·99627	3·25	0·99885
21	97289	56	98953	91	99639	26	99889
22	97358	57	98983	92	99650	27	99892
23	97425	58	99012	93	99661	28	99896
24	97491	59	99040	94	99672	29	99900
25	97555	60	99068	95	99682	30	99903
26	97618	61	99095	96	99692	31	99907
27	97679	62	99121	97	99702	32	99910
28	97739	63	99146	98	99712	33	99913
29	97798	64	99171	2·99	99721	34	99916
30	97855	65	99195	3·00	99730	35	99919
31	97911	66	99219	01	99739	36	99922
32	97966	67	99241	02	99747	37	99925
33	98019	68	99263	03	99755	38	99928
34	98072	69	99285	04	99763	39	99930
35	98123	70	99307	05	99771	40	99933
36	98172	71	99327	06	99779	41	99935
37	98221	72	99347	07	99786	42	99937
38	98269	73	99367	08	99793	43	99940
39	98315	74	99386	09	99800	44	99942
40	98360	75	99404	10	99806	45	99944
41	98405	76	99422	11	99813	46	99946
42	98448	77	99439	12	99819	47	99948
43	98490	78	99456	13	99825	48	99950
44	98531	79	99473	14	99831	49	99952
45	98571	80	99489	15	99837	50	99953
46	98611	81	99505	16	99842	51	99955
47	98649	82	99520	17	99848	52	99957
48	98686	83	99535	18	99853	53	99958
49	98723	84	99549	19	99858	54	99960
50	98758	85	99563	20	99863	55	99961
51	98793	86	99576	21	99867	56	99963
52	98826	87	99590	22	99872	57	99964
53	98859	88	99602	23	99876	58	99966
2·54	0·98891	2·89	0·99615	3·24	0·99880	3·59	0·99967

TABLE 6 (*contd.*)

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
3·60	0·99968	3·70	0·99978	3·80	0·99986	3·90	0·99990
61	99969	71	99979	81	99986	91	99991
62	99971	72	99980	82	99987	92	99991
63	99972	73	99981	83	99987	93	99992
64	99973	74	99982	84	99988	94	99992
65	99974	75	99982	85	99988	95	99992
66	99975	76	99983	86	99989	96	99992
67	99976	77	99984	87	99989	97	99993
68	99977	78	99984	88	99990	98	99993
3·69	0·99978	3·79	0·99985	3·89	0·99990	3·99	0·99993

TABLE 7. FUNCTION $\varphi(x) = \frac{1}{2}\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$
0·00	0·39894	0·20	0·39104	0·40	0·36827	0·60	0·33322
01	39892	21	39024	41	36678	61	33121
02	39886	22	38940	42	36526	62	32918
03	39876	23	38853	43	36371	63	32713
04	39862	24	38762	44	36213	64	32506
05	39844	25	38667	45	36053	65	32297
06	39822	26	38568	46	35889	66	32086
07	39797	27	38466	47	35723	67	31874
08	39767	28	38361	48	35553	68	31659
09	39733	29	38251	49	35381	69	31443
10	39695	30	38139	50	35207	70	31225
11	39654	31	38023	51	35029	71	31006
12	39608	32	37903	52	34849	72	30785
13	39559	33	37780	53	34667	73	30563
14	39505	34	37654	54	34482	74	30339
15	39448	35	37524	55	34294	75	30114
16	39387	36	37391	56	34105	76	29887
17	39322	37	37255	57	33912	77	29659
18	39253	38	37115	58	33718	78	29430
0·19	0·39181	0·39	0·36973	0·59	0·33521	0·79	0·29200

TABLE 7 *(contd.)*

x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$
0·80	0·28969	1·20	0·19419	1·60	0·11092	2·00	0·05399
81	28737	21	19186	61	10915	01	05292
82	28504	22	18954	62	10741	02	05186
83	28269	23	18724	63	10567	03	05082
84	28034	24	18494	64	10396	04	04980
85	27798	25	18265	65	10226	05	04879
86	27562	26	18037	66	10059	06	04780
87	27324	27	17810	67	09893	07	04682
88	27086	28	17585	68	09728	08	04586
89	26848	29	17360	69	09566	09	04491
90	26609	30	17137	70	09405	10	04398
91	26369	31	16915	71	09246	11	04307
92	26129	32	16694	72	09089	12	04217
93	25888	33	16474	73	08933	13	04128
94	25647	34	16256	74	08780	14	04041
95	25406	35	16038	75	08628	15	03955
96	25164	36	15822	76	08478	16	03871
97	24923	37	15608	77	08329	17	03788
98	24681	38	15395	78	08183	18	03706
0.99	24439	39	15183	79	08038	19	03626
1·00	24197	40	14973	80	07895	20	03547
01	23955	41	14764	81	07754	21	03470
02	23713	42	14556	82	07614	22	03394
03	23471	43	14350	83	07477	23	03319
04	23230	44	14146	84	07341	24	03246
05	22988	45	13943	85	07206	25	03174
06	22747	46	13742	86	07074	26	03103
07	22506	47	13542	87	06943	27	03034
08	22265	48	13344	88	06814	28	02965
09	22025	49	13147	89	06687	29	02898
10	21785	50	12952	90	06562	30	02833
11	21546	51	12758	91	06438	31	02768
12	21307	52	12566	92	06316	32	02705
13	21069	53	12376	93	06195	33	02643
14	20831	54	12188	94	06077	34	02582
15	20594	55	12001	95	05959	35	02522
16	20357	56	11816	96	05844	36	02463
17	20121	57	11632	97	05730	2·37	0·02406
18	19886	58	11450	98	05618		
1·19	0·19652	1·59	0·11270	1·99	0·05508		

TABLE 7 *(contd.)*

x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$
2·38	0·02349	2·73	0·00961	3·08	0·00348	3·43	0·00111
39	02294	74	00935	09	00337	44	00107
40	02239	75	00909	10	00327	45	00104
41	02186	76	00885	11	00317	46	00100
42	02134	77	00861	12	00307	47	00097
43	02083	78	00837	13	00298	48	00094
44	02033	79	00814	14	00288	49	00090
45	01984	80	00792	15	00279	50	00087
46	01936	81	00770	16	00271	51	00084
47	01888	82	00748	17	00262	52	00081
48	01842	83	00727	18	00254	53	00079
49	01797	84	00707	19	00246	54	00076
50	01753	85	00687	20	00238	55	00073
51	01709	86	00668	21	00231	56	00071
52	01667	87	00649	22	00224	57	00068
53	01625	88	00631	23	00216	58	00066
54	01585	89	00613	24	00210	59	00063
55	01545	90	00595	25	00203	60	00061
56	01506	91	00578	26	00196	61	00059
57	01468	92	00562	27	00190	62	00057
58	01431	93	00545	28	00184	63	00055
59	01394	94	00530	29	00178	64	00053
60	01358	95	00514	30	00172	65	00051
61	01323	96	00499	31	00167	66	00049
62	01289	97	00485	32	00161	67	00047
63	01256	98	00470	33	00156	68	00046
64	01223	2·99	00457	34	00151	69	00044
65	01191	3·00	00443	35	00146	70	00042
66	01160	01	00430	36	00141	71	00041
67	01130	02	00417	37	00136	72	00039
68	01100	03	00405	38	00132	73	00038
69	01071	04	00393	39	00127	74	00037
70	01042	05	00381	40	00123	75	00035
71	01014	06	00370	41	00119	76	00034
2·72	00987	3·07	0·00358	3·42	0·00115	77	00033
						78	00031
						3·79	0.00030

TABLE 7 *(contd.)*

x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$	x	$\varphi(x)$
3·80	0·00029	3·85	0·00024	3·90	0·00020	3·95	0·00016
81	00028	86	00023	91	00019	96	00016
82	00027	87	00022	92	00018	97	00015
83	00026	88	00021	93	00018	98	00014
84	00025	89	00021	94	00017	3·99	0·00014

4°. Fresnel's Integrals

Fresnel's sine-integrals:

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2 \, dt$$

$$= \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+3}}{(2k+1)!\,(4k+3)},$$

$$S^*(x) = \int_0^x \sin \frac{\pi}{2} t^2 \, dt = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!\,(4k+3)} x^{4k+3}.$$

Fresnel's cosine-integrals:

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2 \, dt = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{(2k)!\,(4k+1)},$$

$$C^*(x) = \int_0^x \cos \frac{\pi}{2} t^2 \, dt = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{2}\right)^{2k}}{(2k)!\,(4k+1)} x^{4k+1}.$$

Relationships between them:

$$S^*(x) = S\left(\sqrt{\frac{\pi}{2}}\, x\right),$$

$$C^*(x) = C\left(\sqrt{\frac{\pi}{2}}\, x\right).$$

TABLE 8. FRESNEL'S INTEGRALS

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2 \, dt, \quad C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2 \, dt$$

x	$S(\sqrt{x})$	$C(\sqrt{x})$	x	$S(\sqrt{x})$	$C(\sqrt{x})$	x	$S(\sqrt{x})$	$C(\sqrt{x})$
0·00	0·00000	0·00000	0·70	0·15040	0·63558	11·0	0·50478	0·38039
02	00075	11283	72	15657	64276	11·5	44781	39515
04	00213	15955	74	16280	64972	12·0	40581	43456
06	00391	19537	76	16908	65646	12·5	38822	48815
08	00602	22553	78	17541	66299	13·0	39827	54251
10	00840	25206	80	18178	66931	13·5	43249	58458
12	01104	27600	82	18820	67542	14·0	48177	60472
14	01391	29796	84	19467	68135	14·5	53374	59887
16	01699	31834	86	20117	68704	15·0	57580	56933
18	02026	33742	88	20771	69256	15·5	59818	52401
20	02372	35540	90	21428	69788	16·0	59613	47431
22	02735	37243	92	22088	70302	16·5	57089	43234
24	03114	38864	94	22751	70796	17·0	52926	40798
26	03509	40410	96	23417	71273	17·5	48175	40659
28	03919	41896	0·98	24085	71731	18·0	43999	42784
30	04342	43310	1·0	24756	72171	18·5	41389	46597
32	04779	44675	1·5	41535	77908	19·0	40934	51133
34	05229	45985	2·0	56285	75330	19·5	42685	55278
36	05692	47252	2·5	66579	67099	20·0	46165	58039
38	06166	48479	3·0	71168	56102	20·5	50487	58785
40	06652	49661	3·5	70018	45205	21·0	54588	57384
42	07149	50804	4·0	64211	36819	21·5	57481	54227
44	07656	51919	4·5	55649	32525	22·0	58494	50117
46	08173	52981	5·0	46594	32846	22·5	57425	46071
48	08700	54019	5·5	39183	37244	23·0	54578	43066
50	09237	55025	6·0	34985	44327	23·5	50682	41808
52	09782	56000	6·5	34710	52220	24·0	46703	42563
54	10336	56946	7·0	38120	59012	24·5	43605	45108
56	10899	57863	7·5	44148	63184	25·0	42122	48788
58	11469	58753	8·0	51201	63930	25·5	42580	52690
60	12047	59616	8·5	57546	61287	26·0	44830	55863
62	12632	60453	9·0	61721	56080	26·5	48293	57552
64	13224	61265	9·5	62857	49689	27·0	52105	57377
66	13823	62053	10·0	60844	43696	27·5	55337	55413
0.68	0·14428	0·62817	10·5	0·56318	0·39509	28·0	0.57214	0·52170

TABLE 8 *(contd.)*

x	$S(\sqrt{x})$	$C(\sqrt{x})$	x	$S(\sqrt{x})$	$C(\sqrt{x})$	x	$S(\sqrt{x})$	$C(\sqrt{x})$
28·5	0·57306	0·48457	36·0	0·50942	0·43421	43·5	0·44676	0·47134
29·0	55621	45183	36·5	47687	43818	44·0	43988	50038
29·5	52600	43136	37·0	45040	45714	44·5	44772	52900
30·0	48997	42791	37·5	43634	48627	45·0	46821	55024
30·5	45697	44203	38·0	43797	51836	45·5	49621	55900
31·0	43497	47002	38·5	45467	54556	46·0	52484	55330
31·5	42013	50484	39·0	48219	56132	46·5	54710	53468
32·0	44060	53794	39·5	51369	56196	47·0	55765	50780
32·5	46634	56131	40·0	54146	54750	47·5	55404	47931
33·0	49987	56941	40·5	55880	52166	48·0	53731	45616
33·5	53293	56051	41·0	56161	49087	48·5	51166	44393
34·0	55749	53703	41·5	54938	46267	49·0	48343	44549
34·5	56771	50488	42·0	52528	44390	49·5	45952	46031
35·0	56131	47201	42·5	49531	43901	50·0	0·44572	0·48466
35·5	0·54009	0·44641	43·0	0·46683	0·44902			

TABLE 9. FRESNEL'S INTEGRALS

$$S^*(x) = \int_0^x \sin\frac{\pi}{2}t^2\,dt, \quad C^*(x) = \int_0^x \cos\frac{\pi}{2}t^2\,dt$$

x	$S^*(x)$	$C^*(x)$	x	$S^*(x)$	$C^*(x)$	x	$S^*(x)$	$C^*(x)$
0·0	0·00000	0·00000	2·0	0·34342	0·48825	4·0	0·42052	0·49843
0·1	00052	10000	2·1	37427	58156	4·1	47580	57370
0·2	00419	19992	2·2	45570	63629	4·2	56320	54172
0·3	01412	29940	2·3	55315	62656	4·3	55400	44944
0·4	03336	39748	2·4	61969	55496	4·4	46227	43833
0·5	06473	49234	2·5	61918	45741	4·5	43427	52603
0·6	11054	58110	2·6	54999	38894	4·6	51619	56724
0·7	17214	65965	2·7	45292	39249	4·7	56715	49143
0·8	24934	72284	2·8	39153	46749	4·8	49675	43380
0·9	33978	76482	2·9	41014	56238	4·9	43507	50016
1·0	43826	77989	3·0	49631	60572	5·0	49919	56363
1·1	53650	76381	3·1	58182	56159	5·1	56239	49978
1·2	62340	71544	3·2	59335	46632	5·2	49688	43889
1·3	68633	63855	3·3	51929	40569	5·3	44047	50779
1·4	71353	54310	3·4	42965	43849	5·4	51403	55723
1·5	69751	44526	3·5	41525	53257	5·5	55368	47842
1·6	63889	36546	3·6	49231	58795	5·6	47004	45171
1·7	54920	32383	3·7	57498	54195	5·7	45953	53846
1·8	45094	33363	3·8	56562	44809	5·8	54605	52984
1·9	0·37335	0·39447	3·9	0·47520	0·42233	5·9	0·51633	0·44859

TABLE 9 *(contd.)*

x	$S^*(x)$	$C^*(x)$	x	$S^*(x)$	$C^*(x)$	x	$S^*(x)$	$C^*(x)$
6·0	0·44696	0·49953	7·0	0·49970	0·54547	8·0	0·46021	0·49980
6·1	51648	54950	7·1	53602	47331	8·1	53204	52275
6·2	53982	46761	7·2	45725	48874	8·2	48588	46384
6·3	45555	47600	7·3	51895	53927	8·3	49323	53775
6·4	49649	54960	7·4	51607	46010	8·4	52428	47091
6·5	54538	48160	7·5	46070	51602	8·5	46534	51418
6·6	46307	46899	7·6	53885	51563	9·0	49986	53537
6·7	49150	54674	7·7	48201	46278	9·5	53100	48729
6·8	54364	48307	7·8	48965	53947	10·0	0·46817	0·49990
6·9	0·46244	0·47323	7·9	0·53234	0·47597			

5°. EULER'S INTEGRALS

Euler's integral of the first kind (beta-function):

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad (x > 0,\ y > 0).$$

Euler's integral of the second kind (gamma-function):

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad (x > 0).$$

The relationship between them:

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

The logarithmic derivative of the gamma-function $\Gamma(x)$:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{d}{dx}\ln\Gamma(x)$$

$$= -\frac{1}{x} - C + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right),$$

where $C = 0{\cdot}57721566$ – Euler's constant.

The logarithmic derivative of the π-function $\pi(x) = \Gamma(x+1) = x\Gamma(x)$:

$$\Psi(x) = \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{d}{dx}\ln\Gamma(x+1) = \psi(x) + \frac{1}{x},$$

$$\Psi(n) = -C + \sum_{k=1}^{n}\frac{1}{k}.$$

TABLE 10. THE GAMMA FUNCTION $\Gamma(x)$

x	0	2	4	6	8
1·00	1·00000	0·99885	0·99771	0·99657	0·99545
01	0·99433	99321	99211	99101	98993
02	98884	98777	98670	98565	98459
03	98355	98251	98147	98046	97945
04	97844	97744	97644	97546	97448
05	97350	97254	97158	97063	96968
06	96874	96781	96689	96597	96506
07	96415	96325	96236	96148	96060
08	95973	95886	95800	95715	95630
09	95546	95463	95380	95298	95216
10	95135	95055	94975	94896	94817
11	94740	94662	94586	94509	94434
12	94359	94285	94211	94138	94065
13	93993	93922	93851	93781	93711
14	93642	93573	93505	93437	93370
15	93304	93238	93173	93108	93044
16	92980	92917	92855	92793	92731
17	92670	92609	92550	92490	92431
18	92373	92315	92258	92201	92114
19	92089	92033	91978	91924	91870
20	91817	61764	91712	91660	91609
21	91558	91507	91457	91408	91359
22	91311	91263	91215	91168	91122
23	91075	91030	90985	90940	90896
24	90852	90809	90766	90724	90682
25	90640	90599	90559	90519	90479
26	90440	90401	90363	90325	90287
27	90250	90214	90178	90142	90107
28	90072	90037	90003	89970	89937
29	89904	89872	89840	89809	89778
30	89747	89717	89687	89658	89629
31	89600	89572	89545	89517	89491
32	89464	89438	89412	89387	89362
33	89338	89314	89290	89267	89244
34	89222	89199	89178	89157	89136
35	89115	89095	89075	89056	89037
36	89018	89000	88982	88965	88948
37	88931	88915	88899	88884	88868
38	88854	88839	88825	88812	88798
1·39	0·88785	0·88773	0·88761	0·88749	0·88737

TABLE 10 *(contd.)*

x	0	2	4	6	8
1·40	0·88726	0·88716	0·88705	0·88695	0·88686
41	88676	88668	88659	88651	88643
42	88636	88629	88622	88615	88609
43	88604	88598	88593	88589	88584
44	88581	88577	88574	88571	88568
45	88566	88564	88563	88562	88561
46	88560	88560	88561	88561	88562
47	88563	88565	88567	88569	88572
49	88575	88578	88582	88586	88590
49	88595	88599	88605	88610	88616
50	88623	88629	88636	88644	88651
51	88659	88667	88676	88685	88694
52	88704	88714	88724	88735	88746
53	88757	88768	88780	88792	88805
54	88818	88831	88844	88858	88872
55	88887	88902	88917	88932	88948
56	88964	88980	88997	89014	89031
57	89049	89067	89085	89104	89123
58	89142	89161	89181	89202	89222
59	89243	89264	89285	89307	89329
60	89352	89374	89397	89421	89444
61	89468	89492	89517	89542	89567
62	89592	89618	89644	89671	89697
63	89724	89752	89779	89807	89836
64	89864	89893	89922	89952	89982
65	90012	90042	90073	90104	90135
66	90167	90199	90231	90264	90296
67	90330	90363	90397	90431	90465
68	90500	90535	90570	90606	90642
69	90678	90714	90752	90789	90826
70	90864	90902	90940	90979	91018
71	91057	91097	91137	91177	91217
72	91258	91299	91341	91382	91424
73	91467	91509	91552	91595	91639
74	91683	91727	91771	91816	91861
75	91906	91952	91998	92044	92091
76	92137	92185	92232	92280	92328
77	92376	92425	92474	92523	92573
78	92623	92673	92723	92774	92825
1·79	0·92877	0·92928	0·92980	0·93033	0·93085

TABLE 10 *(contd.)*

x	0	2	4	6	8
1·80	0·93138	0·93192	0·93245	0·93299	0·93353
81	93408	93462	93517	93573	93629
82	93685	93741	93797	93854	93912
83	93969	94027	94085	94143	94202
84	94261	94321	94380	94440	94501
85	94561	94622	94683	94745	94807
86	94869	94931	94994	95057	95120
87	95184	95248	95312	95377	95442
88	95507	95573	95638	95705	95771
89	95838	95905	95972	96040	96108
90	96177	96245	96314	96384	96453
91	96523	96593	96664	96735	96806
92	96877	96941	97021	97094	97167
93	97240	97313	97387	97461	97535
94	97610	97685	97760	97836	97912
95	97988	98065	98142	98219	98296
96	98374	98452	98531	98610	98689
97	98768	98848	98928	99009	99089
98	99171	99252	99334	99416	99499
99	0·99581	0·99664	0·99748	0·99832	0·99916
2·00	1·00000				

TABLE 11. LOGARITHMS OF THE GAMMA FUNCTION $\log \Gamma(x)$

x	$\log \Gamma(x)$	x	$\log \Gamma(x)$	x	$\log \Gamma(x)$
1·000	0·00000	1·015	$\bar{1}$·99632	1·030	$\bar{1}$·99280
001	$\bar{1}$·99975	016	99608	031	99257
002	99950	017	99584	032	99234
003	99925	018	99560	033	99211
004	99900	019	99536	034	99188
005	99876	020	99513	035	99166
006	99851	021	99489	036	99143
007	99826	022	99466	037	99121
008	99802	023	99442	038	99098
009	99777	024	99419	039	99076
010	99753	025	99395	040	99053
011	99729	026	99372	041	99031
012	99704	027	99349	042	99009
013	99680	028	99326	043	98987
1·014	$\bar{1}$·99656	1·029	$\bar{1}$·99303	1·044	$\bar{1}$·98965

TABLE 11 *(contd.)*

x	$\log \Gamma(x)$	x	$\log \Gamma(x)$	x	$\log \Gamma(x)$
1·045	$\bar{1}$·98943	1·085	$\bar{1}$·98117	1·35	$\bar{1}$·94995
046	98921	086	98098	36	94948
047	98899	087	98079	37	94905
048	98877	088	98059	38	94868
049	98855	089	98040	39	94834
050	98834	090	98021	40	94805
051	98812	091	98002	41	94781
052	98791	092	97983	42	94761
053	98769	093	97964	43	94745
054	98748	094	97946	44	94734
055	98727	095	97927	45	94727
056	98705	096	97908	46	94724
057	98684	097	97890	47	94725
058	98663	098	97871	48	94731
059	98642	099	97853	49	94741
060	98621	10	97834	50	94754
061	98600	11	97653	51	94772
062	98579	12	97478	52	94794
063	98558	13	97310	53	94820
064	98538	14	97147	54	94850
065	98517	15	96990	55	94884
066	98496	16	96830	56	94921
067	98476	17	96694	57	94963
068	98455	18	96554	58	95008
069	98435	19	96421	59	95057
070	98415	20	96292	60	95110
071	98394	21	96169	61	95167
072	98374	22	96052	62	95227
073	98354	23	95940	63	95291
074	98334	24	95833	64	95359
075	98314	25	95732	65	95430
076	98294	26	95636	66	95505
077	98274	27	95545	67	95583
078	98254	28	95459	68	95665
079	98234	29	95378	69	95750
080	98215	30	95302	70	95839
081	98195	31	95231	71	95931
082	98175	32	95165	72	96027
083	98156	33	95104	73	96126
1·084	$\bar{1}$·98137	1·34	$\bar{1}$·95047	1·74	$\bar{1}$·96229

TABLE 11 *(contd.)*

x	$\log \Gamma(x)$	x	$\log \Gamma(x)$	x	$\log \Gamma(x)$
1·75	$\bar{1}$·96335	1·85	$\bar{1}$·97571	1·95	$\bar{1}$·99117
76	96444	86	97712	96	99288
77	96556	87	97856	97	99462
78	96672	88	98004	98	99638
79	96791	89	98154	99	$\bar{1}$·99818
80	96913	90	98307	2·00	0·00000
81	97038	91	98463		
82	97167	92	98622		
83	97298	93	98784		
1·84	$\bar{1}$·97433	1·94	$\bar{1}$·98949		

TABLE 12. THE FUNCTION $\Psi(x) = \dfrac{d}{dx} \ln \Gamma(x+1)$

x	$\Psi(x)$	x	$\Psi(x)$	x	$\Psi(x)$	x	$\Psi(x)$
0·00	−0·5772	0·25	−0·2275	0·50	0·0365	0·75	0·2475
01	5609	26	2155	51	0458	76	2551
02	5448	27	2038	52	0550	77	2626
03	5289	28	1921	53	0642	78	2701
04	5133	29	1806	54	0732	79	2776
05	4978	30	1692	55	0822	80	2850
06	4826	31	1579	56	0911	81	2923
07	4676	32	1467	57	1000	82	2996
08	4528	33	1357	58	1087	83	3069
09	4382	34	1248	59	1174	84	3141
10	4238	35	1139	60	1260	85	3212
11	4095	36	1032	61	1346	86	3283
12	3955	37	0926	62	1431	87	3353
13	3816	38	0821	63	1515	88	3423
14	3679	39	0717	64	1598	89	3493
15	3543	40	0614	65	1681	90	3562
16	3410	41	0512	66	1763	91	3630
17	3277	42	0411	67	1845	92	3699
18	3147	43	0311	68	1926	93	3766
19	3018	44	0211	69	2006	94	3833
20	2890	45	0113	70	2085	95	3900
21	2764	46	−0·0016	71	2165	96	3967
22	2640	47	+0·0081	72	2243	97	4033
23	2517	48	0176	73	2321	98	4098
0·24	−0·2395	0·49	0·0271	0·74	0·2398	0·99	4163
						1·00	0·4228

REFERENCES

1. S. Banach, *Differential and Integral Calculus*. Fizmatgiz, Moscow (1958).
2. A. F. Bermant, *A Course of Mathematical Analysis*, pts. 1 and 2. Pergamon Press, Oxford (1963).
3. A. F. Bermant, *Mapping. Curvilinear Coordinates. Green's Formulae*. Fizmatgiz, Moscow (1958).
4. I. N. Bronshtein and K. A. Semendyayev, *A Guide-Book to Mathematics for Technologists and Engineers*. Pergamon Press, Oxford (1963).
5. C. J. Vallée-Poussin, *Cours d'Analyse Infinitésimale*. Gauthier-Villars, Paris–Louvain (1937–47).
6. G. H. Hardy, *The Integration of Functions of a Single Variable*. Cambridge University Press (1935).
7. V. I. Glivenko, *Stieltjes' Integral*. ONTI, Moscow and Leningrad (1936).
8. E. Goursat, *A Course in Mathematical Analysis*, vol. 1 and 2. Dover Publications, New York (1959).
9. N. M. Gyunter and R. O. Kuzmin, *A Book of Problems in Higher Mathematics*, vols. 1, 2 and 3. M.-L., Gostekhizdat, Moscow-Leningrad.
10. H. B. Dwight, *Tables of Integrals and other Mathematical Formulae*. Macmillan, New York.
11. B. P. Demidovitch, *A Book of Problems and Exercises in Mathematical Analysis*. Fizmatgiz, Moscow (1954).
12. R. Courant, *A Course in Differential and Integral Calculus*, vols. 1 and 2. Blackie, Glasgow (1931).
13. M. A. Lavrentyev and L. A. Lusternik, *Principles of Variational Calculus*, vol. 1, pts. 1 and 2. ONTI, Moscow-Leningrad (1935).
14. E. Landau, *Foundations of Analysis*. Edwards, Ann Arbor.
15. E. Landau, *Differential and Integral Calculus*. Edwards, Ann Arbor.
16. N. N. Lebedev, *Special Functions and their Application*. Gostekhizdat, Moscow (1953).
17. L. A. Lyusternik, Some cubature formulae for double integrals, *Dokl. Akad. Nauk SSSR* 62, 449–452 (1948).
18. E. Madelung, *The Mathematical Apparatus of Physics*.
19. F. M. Morse and G. Feschbach, *Methods of Theoretical Physics*. McGraw-Hill, New York.
20. V. V. Nyemytskii, M. I. Sludskaya and A. N. Cherkasov, *A Course of Mathematical Analysis*, vols. 1 and 2. Gostekhizdat, Moscow-Leningrad (1944).
21. G. P'olya and G. Szegö, *Problems and Theorems in Analysis*, pts. 1 and 2, GONTI, Moscow-Leningrad (1937, 1938).
22. I. M. Rizhik and I. S. Gradstein, *Tables of Integrals, Sums, Series and Products*. Gostekhizdat, Moscow-Leningrad (1951). (English translation in preparation by Pergamon Press, Oxford.)

23. V. I. Smirnov, *A Course of Higher Mathematics*, vols. 1 and 2. Pergamon Press, Oxford (1964).
24. *Tables of Fresnel Integrals*, edited by V. A. Ditkin, published by Academy of Sciences USSR (1953).
25. *Tables of the Integral Sine and Cosine*, edited by V. A. Ditkin, published by the Academy of Sciences USSR (1954).
26. *Tables of the Integral Exponential Function*, edited by V. A. Ditkin, published by the Academy of Sciences USSR (1954).
27. *Tables of Special Functions*, pt. 1, edited by J. N. Shpilrein. G.TTI, Moscow-Leningrad (1933).
28. G. P. Tolstov, *A Course of Mathematical Analysis*, vols. 1 and 2. Gostekhizdat, Moscow (1954, 1957).
29. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, pts. 1 and 2. Cambridge University Press (1934).
30. G. M. Fikhtengol'ts, *A Course of Differential and Integral Calculus*, vols. 1, 2 and 3. Gostekhizdat, Moscow-Leningrad (1948, 1949).
31. G. M. Fikthengol'ts and I. P. Natanson, *Curvilinear and Multiple Integrals*. ONTI, Moscow (1937).
32. B. A. Fuchs and V. I. Levin, *Functions of a Complex Variable and some of their Applications*. Pergamon Press, Oxford (1961).
33. G. H. Hardy, *A Course of Pure Mathematics*. IL, Moscow.
34. A. J. Khinchin, *Eight Lectures in Mathematical Analysis*. Gostekhizdat, Moscow-Leningrad (1955).
35. A. J. Khinchin, *A Short Course of Mathematical Analysis*. Gostekhizdat, Moscow-Leningrad (1955).
36. N. G. Chebotarev, Newton's polygon and its role in the contemporary development of mathematics, *Symposium "I. Newton"*, pp. 29–126 (1943).
37. N. G. Chebotarev, *Theory of Algebraic Functions*. Gostekhizdat, Moscow-Leningrad (1948).
38. E. Césaro, *An Elementary Textbook of Algebraic Analysis and Infinitesimal Calculus*, pt. 1. ONTI, Moscow-Leningrad (1936).
39. E. Césaro, *Elementary Textbook of Algebraic Analysis and Infinitesimal Calculus*, pt. 2. Matezis, Odessa (1914).
40. L. Euler, *An Introduction to the Infinitesimal Calculus*. ONTI (1936).
41. L. Euler, *Differential Calculus*. Gostekhizdat, Moscow-Leningrad (1949).
42. L. Euler, *Integral Calculus*. vols. I, II and III. Gostekhizdat, Fizmatgiz, Moscow, (1957, 1958).
43. E. Janke and F. Emde, *Tables of Functions with Formulae and Curves*, 6th edn. (revised by F. Lösch). McGraw-Hill, New York (1960).

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